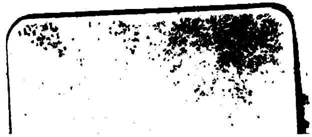

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CAMBRIDGE SCHOOL AND COLLEGE TEXT BOOKS.

ELEMENTARY STATICS.

BY

HARVEY GOODWIN, D.D.

DEAN OF ELY.

NEW EDITION REVISED AND CORRECTED.

CAMBRIDGE:
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1865.

PREFACE TO THE FIRST EDITION.

THE following treatise is not a mere reprint, but may be regarded as a new and modified edition of my Elementary Treatise on Statics. The structure of the book has been considerably changed, and I have given up, in deference to the opinion of several friends, the conversational form which I adopted in my former work for purposes of explanation. Doubtless there were disadvantages attaching to the method which I then employed, but I am bound to say, that in abandoning it I have felt that there were also advantages for the loss of which I am not sure that I have been able to compensate.

No addition of any importance has been made, with the exception of the appendix to Chapter VII, on the Principle of Virtual Velocities. This appendix can be either read or omitted by the student at the option of his tutor.

I have adhered to the division of the subject into *experimental* and *demonstrative* Mechanics; I believe that this mode of treatment is very advantageous, and that even those students who are intending to proceed to the higher branches of Mechanics and to higher modes of treatment will find considerable advantage in regarding

the elementary principles as in a certain sense capable of experimental demonstration, and in knowing mechanical truths not only as the results of mathematical reasoning, but as truths familiarly exhibited to the eye. The excellent apparatus devised by Professor Willis, and referred to in the note of p. 11, makes the experimental treatment of the subject possible and easy for every Lecture-Room; and I think that the duty of dealing with the science of Mechanics in this way, of course in subordination to a genuine mathematical treatment, cannot be pressed too strongly upon all teachers and lecturers.

THE DEANERY, ELY,
May, 1861.

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CHAPTER I.

INTRODUCTORY.

1. IN the following elementary treatise it will be assumed that the reader is acquainted with Plane Geometry as found in Euclid's Elements, with Algebra, and with Plane Trigonometry. It would be possible to gain some knowledge of Mechanics without having first studied the latter two branches of pure Mathematics, but it is so very much to the advantage of the student to postpone his entrance upon mixed Mathematics until he has become familiar with the elements of the subjects just mentioned, that I shall assume him to have adopted this order of study.

2. Having spoken of *pure* and *mixed* Mathematics, let me explain what is meant by the terms, and point out the distinction between these two great branches of science.

Pure Mathematics are concerned with the conceptions of space and number, or space and quantity. Geometry is the science of space. Algebra that of number or quantity. Trigonometry, as usually treated, combines both space and quantity. All the higher branches of pure Mathematics involve the same conceptions, and may in fact be classed either as Geometrical or Algebraical, or as partly Geometrical and partly Algebraical.

Mixed Mathematics is the term used to include all those sciences, in which mathematical reasoning is employed upon conceptions other than those of space and number. For example: Mechanics, or the science of Force; Optics; Hydrostatics; Astronomy.

3. The superior difficulty of this latter class of sciences will be perceived at once. Geometry is made to depend

upon definitions and axioms, the meaning and the truth of which are easily seen. There is no difficulty in understanding what is meant by a triangle, a circle, a parallelogram; the mind easily grants that the whole is greater than its part, that if equals be added to equals the wholes will be equal, that two straight lines cannot include a space, and so on. The definitions and axioms of number are more simple still. But when we wish to reduce the science of force to mathematical calculation, it is not so easy to devise definitions and axioms upon which we can reason: what *is* force? how is its effect to be measured? what principles can we lay down, as the ground upon which our reasoning can be conducted?

4. It will be the purpose of the following pages to answer these questions; that is, to shew how mechanical problems, or problems involving the conception of force, can be reduced to mathematical reasoning. Let me introduce the subject by observing that the science of mechanics divides itself into two great branches. If I throw a ball into the air, it is a mechanical problem to find the path which the ball will describe, the time which will elapse before it strikes the ground, the manner in which it will rebound after striking, the path which it will describe after the rebound, and so on. This would be called a *dynamical* problem, and belongs to the more difficult branch of the subject. There are much easier problems: for example; I take a steelyard and suspend a weight from its shorter arm, and another weight from its longer arm, and it is a mechanical problem to find the proportion of the weights when the arms are given, or the proportion of the arms when the weights are given. This would be called a *statical* problem. The fundamental distinction is, that in the former case we have *motion*, in the latter we have none.

5. In the present treatise we shall be concerned only with the simpler branch of Mechanics, which is called *Statics:* that is, we shall be concerned with problems, in which the forces are so balanced one against another that they do not produce any motion.

It will be observed, however, that although in those problems with which we shall be engaged there be no motion, there is a *tendency* to motion. A book upon a table

does not move, because it is supported by the table; but it has a *tendency* to move, and *would* move, if the table were taken away. If I push against a heavy block of stone, I shall probably not move it, because the friction will be so great as to prevent it from moving; but it has a *tendency* to move, and *would* move, if I were strong enough to move it. Hence *we define force by reference to motion, or tendency to motion;* and we say that,

Any cause which produces or tends to produce motion in a body is called force.

6. This is the formal definition of force, and applies equally to Statics and Dynamics. Now let me say a few words upon the meaning of the term *body*, which occurs in the definition.

We describe all things which are the objects of our senses, as wood, lead, water, air, &c. under the name of *matter*, and any portion of matter whether large or small is called a *body*. If the *body* be inconceivably small it is called a *particle*, and a body may be regarded as made up of particles. Every particle is a body, but all bodies are not necessarily particles. In all cases in which we speak of the action of a force, it is necessary to suppose the existence of a body upon which the action takes place.

7. The next point for our consideration is the method of measuring a force. In statics, as I have said, forces do not produce motion, but counteract the effect of each other so as just not to produce motion. We shall not be able therefore in statics to measure a force by any reference to the motion which it will produce in a given body; we must measure it by comparing it with some known or standard force. Nothing can be more convenient for this purpose than a given *weight*. Take for instance a weight of 1 lb.: if a force will just lift this weight, the force may properly be described as a force of 1 lb. If it will just lift two such weights fastened together, it may be described as a force of 2 lbs.: and so on.

And generally, if a force is just capable of lifting a weight of P lbs., it is called a force of P lbs., or more shortly, a force P; and a force just capable of lifting a weight of Q lbs. is called a force Q. Whenever therefore in this treatise on Statics, the reader finds a force denoted by a letter of the alphabet, as P, Q, R, &c., he will under-

stand that these letters represent numbers, and that the numbers are the numbers of pounds which the forces are respectively capable of just lifting.

8. He will understand also that it is for convenience, and not of necessity, that reference is made to *weight*, as a standard by which to measure force. Instead of taking as the unit of force that force which will just lift a weight of 1 lb., we might take the force which would just break a given beam, or just bend a given spring, but it is manifest that this would not be so convenient a method.

9. In order to describe a force something else must be given beside the number of pounds which it will just lift. Two horses of equal strength move a carriage because they are harnessed in such manner as to pull in parallel directions; if they were harnessed on opposite sides of the carriage, so as to pull in opposite ways, the carriage would not move; and yet they might be exerting precisely the same amount of force as before. Hence to know the real effect of a force, we must know not only its *magnitude*, but also its *direction;* not only the intensity with which it pulls, but the direction in which it tends to make the particle on which it acts move, and in which the particle would begin to move if not hindered.

And not only must we know the magnitude and direction of a force, in order to know its real effect, but it is easy to satisfy ourselves that there is nothing else connected with a force that we can know. If we know the magnitude and direction, then we know the force completely.

10. Hence a very convenient mode of representing forces suggests itself, which will be used continually throughout this treatise.

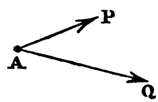
In order to determine or describe a finite straight line, it is necessary that we should know concerning it just these two things and no more, which have been described as necessary in the case of a force. We must know the magnitude of the straight line and its direction, and we then know everything. It is clear therefore that we may take a finite straight line to represent a force: the direction in which the straight line is drawn will shew the direction of the force, and with regard to its magnitude it will only be necessary to agree that a certain length of line shall represent a certain amount of force; as for instance, that 1 inch

shall stand for 1 lb.; and then by measuring the line we shall know how many pounds it is intended to represent.

11. This method of representing forces, which is of infinite use in mechanics, is called representing forces geometrically. I will endeavour to describe it more precisely.

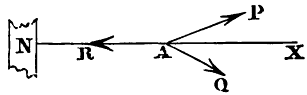
Let AB represent in magnitude and direction a certain force P. Then P stands for P lbs., and if we agree to represent a pound by an inch, AB must contain P inches. Moreover the position of AB on the paper represents P's direction, or (as it is sometimes called) *the line of* P'*s action*. There is nothing however in this representation to shew whether the force is acting from A towards B, or from B towards A. This deficiency in the representation may be supplied by the introduction of an arrow-head, as in the annexed figure, in which the force is supposed to act upon a particle at A, and to tend to make it move from left to right across the paper. Sometimes the symbol which denotes the magnitude of the force is also introduced; thus the annexed figure would represent a force P, tending to make the particle at A move in the direction of the arrow-head.

12. This method may be further applied to represent two or more forces acting upon the same particle. Thus the particle A may be under the action of two forces P and Q, acting in different directions, and we should represent this state of things by such a figure as is here given. And so on for any number of forces.



13. The representation here given of two forces acting on the same particle introduces us to that which is the fundamental problem of Statics. It is manifest that two forces acting on a particle as in the last figure cannot possibly counteract each other, cannot produce equilibrium. Motion would ensue; and such motion cannot be prevented, except by the action of some third force. Of course two equal forces, if they act in opposite directions, can keep a particle at rest, or in equilibrium; but in no other case is this possible; in general there must be *three* forces at least, in order that a particle may be kept by their united

action in equilibrium. Then suppose we have two forces P and Q acting upon a particle A, and suppose that the particle would begin to move in the direction AX; produce AX



backwards to N, and suppose AN to be a string made fast at N; then it is evident that the particle cannot move at all; and therefore it will be possible to find a force R, which acting in the direction AN will counteract the combined effect of the forces P and Q. Thus it is clear that three forces may be found to keep a particle at rest; and that if two forces be given, a third may always be found which shall counteract the effect of the other two. The discovery of the magnitude and direction of this third force, which will counteract the effect of two given forces, is the fundamental problem in Statics.

14. There are two methods of solving this problem. The one by reference to experiment; the other by mathematical demonstration depending upon axioms after the manner of geometry. This latter is the true scientific method; but there is considerable advantage to be derived from treating the subject in the first instance experimentally; the reader will find himself familiarized by the experimental treatment with the new conceptions belonging to the subject, before he has to deal with those conceptions in their more abstract form; he will know in fact distinctly what it is that he is required to prove mathematically, before he is called to the effort of following and understanding the proof.

15. I say *the reader;* but I cannot refrain from taking this opportunity of reminding the student, that he will find *writing* a much more effective mode of study than *reading*. Let him write out from the book several times any difficult proposition, and he will find that he has gained more knowledge of the proposition than he could have gained in a much longer time spent in merely reading it. The method of writing, which appears slow and laborious, is in reality to all, except a very few, an important economy of time and trouble.

16. The next two Chapters deal with statical principles treated experimentally.

CHAPTER II.

EXPERIMENTAL MECHANICS. COMPOSITION AND RESOLUTION OF FORCES WHICH ACT AT ONE POINT, OR ON THE SAME PARTICLE.

1. THE simplest case of forces acting at the same time upon the same particle, is that of two forces acting in such a manner as to keep it at rest.

It is evident that a particle under the action of two forces cannot be at rest unless the two forces be exactly equal in magnitude and opposite in direction. Let us denote by P and Q two forces acting upon a particle in opposite directions, then for equilibrium we must have

$$P = Q \qquad (1),$$
$$\text{or } P - Q = 0 \qquad (2).$$

2. If the forces P and Q be not equal, the greater will preponderate; and the particle will be under exactly the same circumstances as if it were acted upon by the excess of the greater over the smaller force. For instance, if a particle be acted upon by a force of 5 lbs., tending to draw it from left to right across the leaf of this book, and by a force of 3 lbs., tending to draw it from right to left, the particle will be under exactly the same circumstances as if it were acted upon by a force of 2 lbs., tending to draw it from left to right. In this case the force of 2 lbs. is said to be the *resultant* of the two opposite forces 5 lbs. and 3 lbs. More generally, if the two forces P and Q act in opposite directions, and P be the greater, $P - Q$ will be the *resultant* of P and Q, and will tend in the same direction as P. The resultant of these two forces may be denoted by R, and we shall then have

$$P - Q = R \qquad (3).$$

It is evident that if P act upon the particle and tend to draw it in one direction, and Q together with R tend to draw it in the opposite direction, the particle will be at rest.

If P and Q tend to draw the particle in the same direction, and we call R their resultant, we shall have in like manner

$$P + Q = R \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And generally, if a particle be acted upon by any number of forces $P_1\, P_2\, P_3 \ldots$ tending to draw it in one direction, and by any number $Q_1\, Q_2\, Q_3 \ldots$ tending to draw it in the opposite direction, and we call R the resultant, we shall have

$$P_1 + P_2 + P_3 + \ldots - Q_1 - Q_2 - Q_3 - \ldots = R \ldots\ldots (5).$$

3. It is frequently convenient to use a symbol for a force, which shall indicate not only the magnitude of the force, but also the direction in which it acts. Now in Trigonometry we make use of the signs + and − to indicate the directions in which straight lines are drawn, and very great advantage is derived therefrom. If we take a fixed point A, and draw a straight line AB of length a in a given direction, and then draw a straight line AB' of the same length (a) in the exactly opposite direction, we distinguish AB from AB' by calling $AB + a$, and $AB' - a$. It is quite unnecessary to explain to any person, who is acquainted with Trigonometry, the remarkable simplicity and generality which is given to formulæ by this method. Suppose then we adopt a similar convention respecting forces; that is, if A be a particle acted upon by a force P tending to move it from A towards B, let us denote the force by $+P$, and then we can denote by $-P$ an equal force tending to move the particle from A towards B'.

4. We can by this convention enunciate, in a very neat and simple form, a proposition which expresses the rule for finding the resultant of any number of forces, acting upon a particle along the same straight line, but some tending to move it in one direction along the straight line, and others to move it in the exactly opposite direction; for we may say that

The resultant of any number of forces having the same line of action is the algebraical sum of the forces.

Or if we denote by $P_1\, P_2\, P_3 \ldots$ any number of forces acting in the same line, and by R their resultant, and if $P_1\, P_2\, P_3 \ldots$ represent positive or negative quantities as the case may be, we shall have

$$P_1 + P_2 + P_3 + \ldots = R \;\ldots\ldots\ldots\ldots (6).$$

5. We may further make use of the convention concerning positive and negative forces to enunciate the general condition of equilibrium of any number of forces acting upon a particle and having the same line of action; for we may say, that under such circumstances *the particle will be at rest if the algebraical sum of the forces be zero.*

6. The term *resultant,* which we have used in this very simple case, is one of much more general application. Whatever forces may act on a particle and in whatever directions, they will be equivalent to one single force; for if they be not such as to keep the particle at rest, or to produce equilibrium, the particle will begin to move in a certain direction, and it can be prevented from moving by a single force acting upon it in the exactly opposite direction; call the force which is just sufficient for this purpose R, then R is in equilibrium with the whole system of forces; but so it would be with a force R applied in the direction in which the particle would begin to move; consequently the whole system of forces is equivalent to one single force R, acting in that direction in which the particle would begin to move. And R is therefore termed *the resultant of the system of forces.*

7. The general problem of Statics may be said to be, to find the resultant of any system of forces. The problem is much simplified by supposing the lines of action of all the forces to lie in the same plane, which for clearness of conception we will suppose to be the plane of the paper; and to this limited case we shall confine ourselves. It will be found, however, that notwithstanding this limitation the results at which we shall arrive will be applicable to a very large class of interesting questions. Our first step must be to solve this problem: Given the magnitude and direction of two forces (P and Q) which act upon a particle, to determine their resultant (R). Or, which is the same thing,

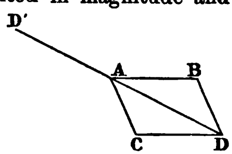
Given that three forces acting in the same plane upon a particle keep it at rest, to find the relations which subsist amongst the magnitudes and directions of the three forces.

8. Now in considering the case in which a particle was acted upon by forces having the same line of action, we were able at once to deduce our results by reference to the simplest principles: but the problem with which we are now concerned does not admit of so easy a solution. There are (as I have already observed in the preceding chapter) two modes of solving it; we may either have recourse to experiment, or we may demonstrate it mathematically by means of certain axioms concerning force. At present we shall be concerned with the former method.

9. We might commence with some experiments for the purpose of determining the relations, which the resultant of two forces given in magnitude and direction will bear to the forces themselves; but it will be more simple to enunciate the result at which mathematicians have arrived, in the form of a theorem, and then shew how the theorem may be experimentally verified. This theorem is usually called the *Parallelogram of Forces*, and may be enunciated as follows:

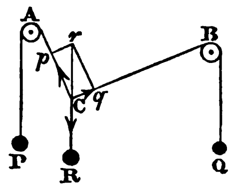
If two forces acting upon a particle be represented in magnitude and direction by two straight lines drawn from the particle, then the diagonal of the parallelogram described upon these two straight lines will represent the resultant in magnitude and also in direction.

Suppose, for instance, that A is the particle, and let it be acted upon by a force represented in magnitude and direction by AB, and by another force represented in magnitude and direction by AC; complete the parallelogram $ACDB$; then the resultant force will act in the direction AD, and will be represented by AD in magnitude. So that if we produce AD backwards to D', and make AD' equal to AD, then AB, AC, AD', will represent three forces in equilibrium upon the particle A.



10. We shall now explain a method of proving by experiment the truth of this theorem.

Let A and B be two small wheels turning upon horizontal pivots inserted into a vertical board. The wheels must be carefully constructed, so as to avoid friction as much as possible. Let P, Q, R be three weights attached to the extremities of three silk cords, the other extremities of which are all knotted together in one point C*.



Now let the cords supporting two of the weights, as P and Q, be made to rest upon the wheels A and B, as in the figure, and let the weight R hang from the point C. Then it will not be difficult to arrange the system in such a manner that it shall rest as represented in the figure; we have therefore the point C kept at rest by three forces acting in the direction of the three cords.

And these three forces will be measured by the three weights P, Q, R: for the effect of the wheels A and B is (neglecting any effect arising from friction) merely to change the direction of the strings; that is, the force which must be applied at C in the direction AC in order to support the weight P must be a force whose measure is P. The point C is therefore kept at rest by three forces P, Q, R acting in the directions of the three cords, which support the weights P, Q, and R respectively.

Along CA measure a line Cp containing as many inches as there are ounces in the weight P; and along CB a line Cq containing as many inches as there are ounces in Q; complete the parallelogram $Cprq$; and join Cr. Then it will be found that Cr is sensibly in the direction of RC produced, that is, vertical; and if Cr be measured, it will be found to contain as many inches as there are ounces in the weight R.

By these means, therefore, the Parallelogram of Forces can be proved, so far as such a proposition can be proved experimentally: the more accurately the experiments are made, and the more they are varied in their circumstances,

* For this and for all other experimental illustrations of Statical principles I would recommend the apparatus devised by Professor Willis, and which may be obtained from Messrs Elliott, Brothers, Strand, London; or from Messrs Rigg, of Chester.

so much the more certain will they render the truth of the proposition.

11. Let us now take a few illustrative examples.

Ex. 1. Suppose $P=3$ lbs., and $Q=4$ lbs., and the angle between them to be a right angle. Then it will be seen that the lines representing P, Q, and R form a right-angled triangle, and therefore by Euclid I. 47,

$$R^2=P^2+Q^2=9+16=25,$$
$$\therefore\ R=5.$$

The angle which R makes with P is that whose cosine is $\frac{3}{5}$ or $\cdot 6$, which will be found to be $53^0\ 52'\ 11''$ nearly.

Ex. 2. Two forces act on a particle, the angle between their directions being 60^0; to find the magnitude of their resultant.

Let P and Q represent the forces; R the resultant; then in the figure of page 10,

$$AB=P,\ BD=Q,\ AD=R,\ ABD=180^0-BAC=120^0;$$
$$\therefore\ R^2=P^2+Q^2-2PQ\cos 120^0$$
$$=P^2+Q^2+PQ, \text{ since } \cos 120^0=-\tfrac{1}{2}.$$

Ex. 3. Suppose in the preceding example $P=2$lbs., $Q=3$lbs.

$$\therefore\ R^2=4+9+6=19;$$
$$\therefore\ R=\sqrt{19}=4\cdot 3589 \text{ lbs.}$$

Ex. 4. To find the direction of R in the preceding example,

$$\sin BAD=\frac{BD}{AD}\sin ABD=\frac{3}{\sqrt{19}}\sin 120^0=\frac{3\sqrt{3}}{2\sqrt{19}},$$
$$\therefore\ BAD=36^0\ 35'\ 12'',$$

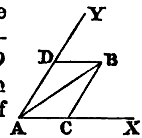
as will be found by help of a logarithmic table of sines.

12. When by means of the parallelogram of forces we find a single force which is equivalent to two others, we are said to *compound* those forces. And the parallelogram of forces may be described as the *rule for the composition of forces.*

Conversely we can by the same proposition find two forces, which shall be equivalent to any one given force; when we do this, we are said to *resolve* the force into two others, or into two *components;* so that the parallelogram of forces is sometimes spoken of as the *rule for the composition and resolution of forces.*

If two component forces be given in magnitude and direction, it is a determinate problem to find the magnitude

and direction of their resultant; but if the magnitude and direction of one force be given, it is not a determinate problem to find the two components of which it is the resultant, since there are an infinite number of pairs of forces, from the composition of which the given force may be conceived to have resulted. If, however, it be required to find two components in given directions from which a given force shall result, the problem can be solved. Thus let AB represent the given force in magnitude and direction; AX, AY the given directions of the components. Draw BC, BD parallel to AY, AX respectively; then AC, AD will represent the magnitudes of the two components required.



Or, again, if one of the components be given in magnitude and direction, the other can be found. Thus, let AB be a given force as before, and let AC be one of its components; join BC, and complete the parallelogram $ADBC$, then AD will be the other component.

Ex. 1. A force of 6 lbs. is the resultant of two forces, with the direction of which it makes respectively angles of 30° and 45°: required the magnitude of each component.

Referring to the preceding figure we shall have

$$AB=6,\ BAC=30^0,\ ABC=BAD=45^0;$$

$$\therefore AC=6\times\frac{\sin 45^0}{\sin 75^0}=4\cdot3923 \text{ lbs.}$$

$$AD=BC=6\,\frac{\sin 30^0}{\sin 75^0}=3\cdot1058 \text{ lbs.}$$

Ex. 2. Let the resultant be 10 lbs., and let one of the components be 7 lbs., and make with the resultant an angle of 60°, to find the other component.

Referring to the same figure, we have in the triangle ABC

$$AC=10,\ AC=7,\ BAC=60^0, \text{ to find } BC;$$

$$\therefore BC^2=10^2+7^2-2\times10\times7\cos 60^0$$

$$=149-70, \text{ since } \cos 60^0=\tfrac{1}{2}$$

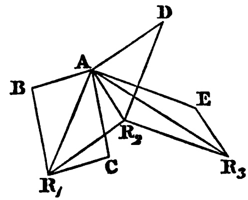
$$=79;$$

$$\therefore BC=\sqrt{79}=8\cdot8882 \text{ lbs.}$$

13. It will further appear that the parallelogram of forces enables us to find the resultant of any number of forces which act upon a single particle. For we have only

to take two, and find their resultant by the rule; then we can compound this resultant with the third; the next resultant with the fourth; and so on for any number.

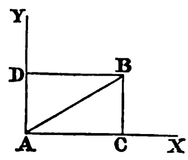
To represent this geometrically; let AB, AC, AD, AE, represent any forces acting on the particle A. Complete the parallelogram ABR_1C, and AR_1 will represent the resultant of AB and AC. Complete the parallelogram AR_1R_2D, and AR_2 will represent the resultant of AR_1 and AD, that is, of AB, AC and AD. Lastly, complete the parallelogram AR_2R_3E, and AR_3 will represent the resultant of AR_2 and AE, that is, of AB, AC, AD and AE.



It is evident that this process applies equally well, whether the lines of action of the forces be all in the same plane or not.

We might, by means of the process here described, actually calculate the magnitude and direction of the resultant of any number of forces acting at one point; the method however would not be convenient, and we shall hereafter explain a process for finding the resultant, the same in principle, but much more easy of application.

14. It has been observed, that a force may be resolved into two others in an infinite number of ways; there is, however, one mode of resolution which deserves particular attention. I refer to the case in which a force is resolved into two components in directions perpendicular to each other. Let AB represent the given force; and AX, AY two directions at right angles to each other. From B let fall the perpendiculars BC, BD on AX, AY respectively; then since $ACBD$ is a parallelogram, AC, AD represent the components of the force in the directions AX and AY. Denote the angle BAC by θ; and let R be the original force, X and Y its two components. Then since

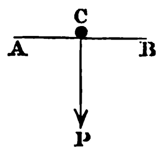


$$AC = AB \cos\theta, \text{ and } AD = AB \sin\theta,$$

we have $\quad X = R\cos\theta, \quad$ and $\quad Y = R\sin\theta.$

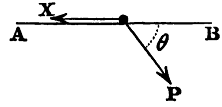
Now the peculiarity of this case consists in this, that *no force has any tendency to produce motion,* in other words, it has no effect, *in the direction perpendicular to its own.* This is evident from the very nature of force; but the truth of it is seen at once from such a consideration as this, that a body placed upon a horizontal table will be at rest, however smooth the table may be. The body is acted upon by a force, namely, its own weight; but this tends to draw it downwards, and the table, being horizontal, will not admit of any motion downwards; consequently the body does not move at all. The pressure downwards is counteracted by the upward pressure of the table, and these two are exactly equal and opposite. The effect of the body's weight therefore is to produce a pressure upon the table, which is measured by the weight; but it has no tendency to produce motion upon the table, that is, there is no component in the direction parallel to the surface of the table, or perpendicular to the direction in which the weight acts.

The same thing may be seen as follows. Let C be a ball having a string attached to it, and let AB be a smooth plane having in it a rectilinear slit through which the string can pass. Then let a force P act at the extremity of the string; if the string be exactly perpendicular to the plane, it is obvious that there will be no motion;



but if otherwise, the ball will move along the plane in the direction of the slit. If the string be perpendicular to the plane, the force P will produce a pressure P upon the plane, and will have no component parallel to the plane or perpendicular to the string.

Suppose however that the direction of the string is not perpendicular to the plane; then motion may be prevented by a force acting along the plane; what will be the magnitude of this force? The preceding investigation will



enable us to determine this. For let θ be the acute angle which the direction of the string makes with the plane; then P may be resolved into two components, $P\cos\theta$ parallel to the plane, and $P\sin\theta$ perpendicular to it; let

X be the force which would prevent the ball from moving; then it is evident that we must have

$$X = P \cos \theta.$$

And the component $P \sin \theta$ will produce a pressure upon the plane, having $P \sin \theta$ for its measure.

The component $P \cos \theta$ may be termed the *resolved part* of P parallel to the plane; and the preceding explanation will enable us to form a very distinct notion of what is meant by the *resolved part* of a force in any given direction; for it is measured by the force which is necessary to prevent the given force from producing motion in that direction. And we have this general rule, of which it is impossible to overestimate the importance, *To find the resolved part of a force in any given direction, multiply the expression for the force by the cosine of the angle between the given direction and that of the given force.*

Suppose, for instance, that in the preceding figure, $P = 4$ lbs., and $\theta = 60^0$; then the resolved part of P along the plane, that is, the force necessary to prevent motion along the plane $= 4 \times \cos 60^0 = 2$ lbs. And the pressure upon the plane

$$= 4 \sin 60 = 2\sqrt{3} \text{ lbs.}$$

Again, suppose a horse is drawing a weight up a hill; and for distinctness let the weight be 1 ton, and let the inclination of the road to the horizon be 4^0; then the resolved part of the weight parallel to the road will be $1 \times \sin 4^0$, or 156·25 lbs.; and if we neglect the resistance to motion arising from the roughness of the road, this will be the measure of the effort which must be made by the horse in order just to drag the load. Practically this resistance may by no means be omitted; it is called *friction*, and will be considered hereafter; but theoretically, that is, supposing the road to be perfectly smooth, the horse would have to exert such a force as would just lift 156·25 lbs. And the resolved part of the weight perpendicular to, and therefore supported by, the plane will be

$$1 \text{ ton} \times \cos 4^0, \text{ or } 2234{\cdot}5 \text{ lbs.}$$

15. We shall return to the subject of the Composition and Resolution of Forces, when we have demonstrated the Parallelogram of Forces independently of experiment.

EXAMINATION UPON CHAPTER II.

1. Define *force*, and explain how force is measured in Statics.

2. Distinguish between *Statics* and *Dynamics*.

3. Explain the method of representing forces by straight lines.

4. Enunciate the parallelogram of forces, and shew how it may be proved experimentally.

5. Two equal forces act upon a point, and the angle between their directions is 60^0; find the magnitude and direction of the resultant.

6. A particle in the centre of an equilateral triangle is urged towards two of the angular points by forces each equal to 3 lbs.; find the force which must tend to the third angular point in order that the particle may be at rest.

7. A particle is acted upon by a horizontal force of 3 lbs. and a vertical force of 4 lbs.; find the direction and magnitude of the resultant force.

8. If two forces act at right angles to each other, they and their resultant are proportional to the sides of a right-angled triangle.

9. Three forces measured by 3, 4, and 5 pounds respectively keep a particle in equilibrium; determine the angles at which they act.

10. The resultant of two forces acting at right angles to each other is double the smaller of the two; find its direction.

11. The resolved part of a force in any direction is found by multiplying the expression for the force by the cosine of the angle between the direction of the force and the given direction.

12. A weight of 112 lbs. rests upon an inclined plane making an angle of 80^0 with the horizon: what is the resolved part of the weight in the direction of the plane? and what the resolved part perpendicular to it?

13. Is it possible for three forces in the proportion of 3, 7, and 11, to be in equilibrium when acting upon a point?

14. Three forces act at a point, and their directions make angles of 120^0 with each other: shew that the three forces are equal.

15. The resultant of two forces which act at right angles to each other is 1 cwt.; and the direction of the resultant divides the right angle in two of 18^0 and 72^0 respectively: find the components in lbs.

16. If two forces P and Q acting upon a particle have a resultant R, and θ be the angle between the directions of P and Q, then

$$R^2 = P^2 + Q^2 + 2PQ \cos\theta.$$

17. Three forces of 2 lbs. 7 lbs. and 8 lbs. are in equilibrium; find the angle between the directions of each two of the forces.

18. The square of the line representing the resultant of two forces is greater or less than the sum of the squares of the lines representing the components, according as the angle between the components is acute or obtuse.

19. The magnitude of two forces being 25 lbs. and 36 lbs., and the angle between their directions 72^0; find the magnitude and direction of the resultant.

20. The angle between two forces is 60^0, and the resultant divides the angle in the ratio of 1 : 3; find the ratio of the components.

21. If P, Q, R be three forces which produce equilibrium upon a point, and if (PQ) denote the angle between the directions of P and Q, then

$$P : Q : R :: \sin(QR) : \sin(RP) : \sin(PQ).$$

22. Three forces acting at a point are in equilibrium, their directions making with each other the angles 60^0, 135^0, and 165^0 respectively; find the ratios of the forces.

23. If two forces acting on a particle be in the ratio of 3 : 4, find the angle between their directions when the resultant is a mean proportional between them.

24. The resultant of two forces cannot be an arithmetical mean between them, if one of the forces be more than three times as great as the other.

25. If three forces be in equilibrium, any two must be together greater than the third.

CHAPTER III.

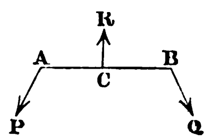
EXPERIMENTAL MECHANICS. THE PRINCIPLE OF THE LEVER.

1. IN the preceding chapter we have treated of the Composition and Resolution of Forces which act all at one point, or of the conditions under which a single particle can remain at rest when two or more forces are acting upon it. We shall now take the case of a body of any magnitude, which is acted upon by various forces, and shall endeavour to discover, by experiment, the conditions which must be satisfied in order that a body of this kind may be in equilibrium. But the problem in this form would be too complicated; and we shall therefore take a simple case, from which afterwards it will not be difficult to pass to the more general.

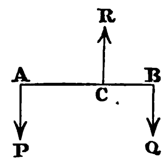
Our first simplification shall be this: instead of considering the conditions of equilibrium of a body of any form, we will take the simplest form of body possible, that is, a *straight rod*, which, however, we shall suppose to be so strong as not to bend under the action of any forces applied to it.

Our second simplification shall be as follows: instead of considering the case of any forces whatever acting upon this stiff or *rigid* rod, we will suppose it to be acted upon by only *three*, and we will suppose the lines of action of these forces to lie all in one plane.

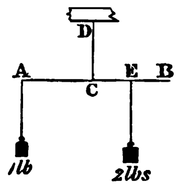
The system may then be represented as in the figure; AB is the rod; P, Q, R are three forces, whose directions lie in the plane of the paper, acting at three points A, B, and C respectively; the problem is to determine under what conditions the rod AB will be at rest.



There is yet one further simplification which we can make; and this consists in supposing the three forces P, Q, R to be parallel in direction, and that direction perpendicular to AB; and the system will then be represented by the figure; in which it will be observed, that we have represented one of the forces acting in a direction diametrically opposite to that of the other two, as must evidently be the case, since equilibrium could not subsist if the forces all tended in the same direction.



2. The problem is now in a form convenient for experiment*; let a rod AB be suspended by a string, one extremity of which is attached to the middle point C of the rod, and the other to some fixed point of support D; now let any given weight, as a weight of 1 lb., for instance, be suspended by a string from the extremity A, and let some larger weight be attached to another string which by means of a ring, or otherwise, is capable of being suspended from any point of the arm BC; then it will be found that if this second weight be suspended from the extremity B, B will descend until the three strings and the rod are all vertical, and by making the point of suspension more and more near to C, we shall come at last to a point from which, if the larger weight be suspended, the rod if placed in a horizontal position will remain so.



For instance, if the larger weight be 2 lbs., it will be found that the point E, from which it must be suspended, is half-way between B and C; and in general, CE will be the same fraction of BC or AC that the smaller weight is of the larger; so that if we have two weights P and Q hanging by strings from two points A and E, and if $CA=p$, and $CE=q$, we shall have

$$P : Q :: q : p;$$

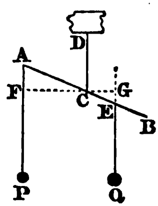
$$\text{or, } P.p=Q.q.$$

If we call the product of the weight P and the perpen-

* See Note at the foot of page 11.

dicular upon its string from C the *moment* of P about C, then we may express the above relation by saying that *the moments of* P *and* Q *about* C *are equal.*

3. It will be found that if this condition between the two weights and the distances of their points of suspension from C be satisfied, the system will remain at rest when C is not horizontal; in this case if we drop the perpendiculars CF, CG from C upon the strings supporting P and Q respectively, we have by similar triangles



$$CF : CA :: CG : CE;$$
$$\therefore P \times CF = Q \times CG,$$

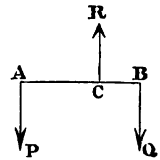
or the moments of P and Q about C are still equal.

4. In this experimental investigation we have hitherto spoken only of the *two* weights suspended from A and E: the third force acting on the rod is the force exerted by the string CD; and what will be the magnitude of this force? It will evidently be measured by the weight which it supports, *i.e.* the two weights P and Q and the weight of the rod; if we omit the weight of the rod, then the force exerted by the string CD, or its *tension* (as it is usually called), will be upwards, and will be equal to $P+Q$.

5. In considering the problem theoretically it is convenient to conceive of the rod AB as having no weight, in order that we may confine our minds to the two downward forces P and Q, and the upward tension of the string $P+Q$, but practically of course AB has weight, though it may be very small; it was in order to avoid the effect of this weight that we directed the rod to be suspended by its *middle* point C, for if this be done it is clear that the extremity A will have no more tendency to descend than the extremity B, and therefore the weights P and Q twist the rod precisely as they would if it had actually no weight.

6. The weights P and Q, which we have been considering, exert forces upon the rod AB in the directions of the strings by which they are suspended; now it is easy to prove by observation that these strings are parallel, at least that they are *sensibly* parallel; also the string by which AB is supported exerts a force in its own direction, which is

parallel to that of the other two strings; hence the preceding experimental investigation teaches us the conditions, under which three parallel forces can be in equilibrium, when acting upon a straight rod. Let P, Q, R be the forces acting upon the rod AB, as in the figure, at the points A, B, and C respectively; then we must have



$$P+Q=R \quad \dots\dots\dots\dots\dots \quad (1),$$

$$\text{and } P\times AC=Q\times BC \quad \dots\dots\dots \quad (2).$$

7. The latter of these two conditions may be put under another form; for if we write $R-Q$ instead of P in (2), we have

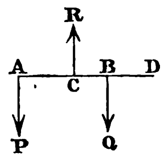
$$(R-Q)\,AC=Q\times BC,$$

$$\text{or } R\times AC=Q\,(AC+BC)=Q\times AB;$$

i.e. the *moments* of R and Q about A are equal.

In like manner it may be shewn, that the moments of P and R about B are equal.

8. And still more generally if we produce AB to any point D, we have



$$\begin{aligned} &P\times AD+Q\times BD \\ &=P\times(AC+CD) \\ &+Q\times(CD-BC) \\ &=(P+Q)\,CD+P\times AC-Q\times BC \\ &=R\times CD \text{ by (1) and (2), (Art. 6).} \end{aligned}$$

i.e. the moment of R about D is equal to the sum of the moments of P and Q about the same point. Now it is evident from inspection, that if we conceive AD to be a rod capable of twisting about one extremity D, and acted upon by three forces P, Q, and R, as in the figure, then P and Q tend to twist it one way and R tends to twist it in the other; hence we may say, that *there will be equilibrium if the moment of the force which tends to twist the rod in one direction be equal to the moments of those tending to twist it in the other.*

9. In the preceding Articles we have been dealing with a new kind of quantity, namely, *the moment of a force about a given point*, and the nature of this new kind of

quantity requires perhaps some further explanation. The moment of a force about a given point is defined to be the product of the force (or the weight which measures it) and the perpendicular upon the direction of the force from the given point. But what is the meaning of the product of a force and the perpendicular upon it? how can a *force* and a *line* be multiplied together?

There is no difficulty in answering these questions, if it be remembered that what we really multiply together are the abstract numbers, which denote the force and the line respectively according to a conventional method of representing forces and lines. Thus, 3 denoting a force may stand for 3 lbs., and 3 denoting a line may stand for 3 feet, and if we multiply together 3 which stands for 3 lbs. and 3 which stands for 3 feet, we shall obtain the number 9, which will stand neither for pounds nor feet, but for something else which we have agreed to denote by the term *moment.* That there is need of such a term is manifest, for we have seen that when a weight is suspended as in Art. 2, its effect in supporting another weight suspended from some point on the opposite side of C does not depend merely upon its own magnitude, nor merely upon the distance of its point of suspension from C, but upon the two jointly; this effect then, which requires to be denoted in some way, we call the *moment* of the weight about C. And we reduce moments to a unit of measurement precisely upon the same principle as forces, lines, areas, &c. are reduced to units of measurement. Suppose we take as the unit of moment the moment of a weight of 1 lb. acting upon an *arm* of 1 foot; in other words, in the figure of Art. 2, let $AC = 1$ foot; then the moments of other weights will be properly measured by the product of the number representing the weight and the number representing the length of the arm upon which the weight acts.

10. We may express the condition given in Art. 8 still more conveniently, by introducing the use of the signs *plus* and *minus* to indicate the tendency of forces to twist the rod in opposite directions.

For if we call the moments of those forces which tend to twist the rod in one direction *positive*, and those which tend to twist it in the opposite direction *negative*, then it will be seen that the condition referred to may be expressed by

saying, that *the algebraical sum of the moments of the three forces about any point in the direction of the rod is zero.*

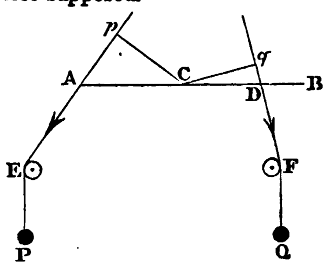
11. And it may be remarked, that in like manner the condition (1), (Art. 6), may be expressed by saying, that the *algebraical sum of the three forces is zero.*

Hence the complete statement of the conditions of equilibrium of three parallel forces acting upon a rod will be as follows,

the algebraical sum of the forces $=0$ .... (1),

the algebraical sum of the moments of the forces about any point $=0$ .... (2).

12. It may be observed that the experimental method of proof used in this chapter might be applied to investigate the condition of equilibrium, when two forces act upon a rod of which one point is fixed but not in directions perpendicular to the rod, as hitherto supposed.



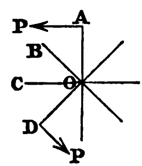
For let AB be a rod moveable about a horizontal pivot through its middle point C; E, F two small wheels turning freely about their centres; and let two silk cords, attached to the rod at two points A and D, pass over E and F and support two weights P and Q. Suppose the magnitudes of P and Q to be so adjusted that AB shall be horizontal; then if we suppose the directions of the cords EA, FD produced, and Cp, Cq drawn from C perpendicular to them, it will be found that

$$P : Q :: Cq : Cp \text{ .... (1)},$$
$$\text{or } P \times Cp = Q \times Cq \text{ .... (2)}.$$

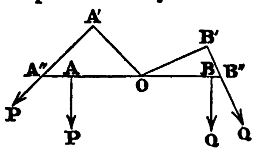
Now the cords exert at A and D forces which are measured respectively by the weights P and Q; hence the proportion (1) shews, that the forces which keep the rod in equilibrium are inversely proportional to the perpendiculars upon them from the fixed point C; or the equation (2) which is equivalent to the proportion (1), shews that the *moments* of the forces about C must be equal.

13. This experimental demonstration however is not worthy of much attention, partly because the simpler case, in which the forces are weights acting in directions perpendicular to the rod, will answer our purpose at present, and partly because the truth of the more general case may be seen without much difficulty to follow from that of the more simple.

For let AO, BO, CO, &c. be any number of equal spokes of a wheel: then it is evident that the effect of a force in turning the wheel will be precisely the same to whichever of the spokes it is applied. For instance, the force P applied at A perpendicularly to OA will have the same effect as if it were applied at D perpendicularly to OP.



Now take a rod AB, without weight, and having the point O in it fixed, and let it be kept at rest by the two forces P and Q acting at A and B perpendicularly to the rod. Through O draw OA' equal to OA, and OB' equal to OB, each making any angle with AB; then if OA', OB' be regarded as rigidly joined together at O, like the spokes of a wheel, it is manifest from what has been just now said, that the forces P and Q acting at A' and B' perpendicularly to OA' and OB' respectively will keep $A'OB'$ at rest.



Again, let $A'P$ intersect OA produced in A'', and $B'Q$ intersect OB produced in B''; then if we conceive of the figure $A''A'OB'B''$ as one rigid board, without weight, capable of turning about O, and suppose the forces P and Q to act by means of strings, the directions of which coincide with $A'A''$ and $B'B''$ respectively, it is evident that the effect will be the same at whatever point of $A'A''$ and $B'B''$ we suppose a tack to be driven through the strings so as to attach them to the board. Now if we suppose the string $A'A''P$ to be attached at the point A' we have the force P acting at the extremity of OA', and if we suppose it attached at A'' we have the force P acting obliquely at the extremity of OA''; hence the effect of P acting at A'' obliquely, as in the figure, is the same as that of P acting at A perpendicularly to OA'. Hence we conclude that P and Q acting

obliquely, as in the figure, at the extremities of a rod AB, having the point O fixed, will produce equilibrium provided their *moments about O are equal.*

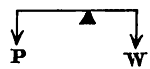
14. We shall defer the further generalization of the conditions of equilibrium investigated in this Chapter, that is, their extension to any number of forces and to forces acting in any directions, to the Chapter in which we investigate the properties of moments of forces theoretically; the remainder of the present we will devote to the consideration of the problem of three forces acting upon a rod, exhibited under a somewhat different form.

15. Def. A rod capable of turning about a fixed point in its length is called a *lever;* and the fixed point is called the *fulcrum.*

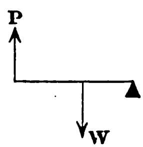
16. Def. If the lever be horizontal and a *weight* W be suspended from any point in its length, the lever may be sustained in a horizontal position by a certain force P acting at some other point, and tending vertically upwards or downwards according to circumstances. The force P which is required to maintain the equilibrium is called the *power.*

17. We may distinguish three classes of lever.

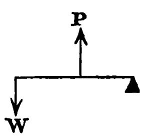
First. Suppose the *fulcrum* to lie between the *power* and the *weight.* This is called a *lever of the first kind.*



Secondly. Suppose the *weight* to act between the *fulcrum* and the *power.* This is called a *lever of the second kind.*



Thirdly. Suppose the *power* to act between the *weight* and the *fulcrum.* This is called a *lever of the third kind.*



18. Now it will be readily perceived, that although in all these cases we have spoken of only two forces, the *power* and the *weight,* as acting upon the lever, yet in reality there must be and are three forces. What is the third? It is supplied by the pressure upon the fulcrum. In the first

case this pressure will be the sum of the power and the weight; in the second and third it will be their difference. Or if we call the pressure on the fulcrum R, we shall have

for the lever of the first kind $R = P + W$,

.... second... $R = W - P$,

.... third ... $R = P - W$.

But there will be this distinction between the pressure on the fulcrum in the second case and the third, namely, that in the second (as in the first) the pressure is upwards, in the third it is a pressure downwards.

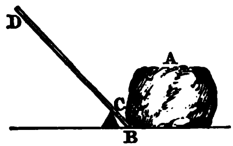
19. When we regard the lever thus, we see that all three cases are instances of the equilibrium of three parallel forces, and that the conditions of equilibrium will be those which have been already investigated. In each lever therefore we must have

moment of power about fulcrum = moment of weight.

If then we suppose that we have a lever with a given fulcrum, and a given weight W suspended at a given point, we can at once determine the point at which a given power P must act in order to be in equilibrium with W; or if the point of application (or, as it is sometimes expressed, *the length of P's arm*) be given, we can calculate P.

20. And the following conclusions will at once appear to be true.

(1) In the case of the lever of the *first* kind we may make P as small as we please, if we increase the arm at which it acts in the same proportion that we diminish P. That is, a weight of any magnitude may be supported by any small force, if we give to this force a sufficient length of arm. Thus a weight of 100 lbs. suspended at a distance of 1 ft. from the fulcrum, may be sustained by a weight of 1 lb. suspended at the other extremity of the lever, provided that that extremity be 100 ft. from the fulcrum. And this points out to us the great advantage which may be gained in practice, by applying a force through the medium of a lever of this kind. Suppose for instance A to be a fragment of rock, a block of wood, or any other great weight which it is required to raise; let BD be a strong bar of iron or



wood, and let one extremity B be inserted just under the weight A, and let the bar be made to rest against a support at C, not far from the extremity B; then a comparatively small force applied at D will enable us to move A. This is the most ordinary application of the lever to common purposes, and probably every one is familiar with examples; in fact, we make use of a lever of this kind every time that we rest the poker upon a bar to stir the fire; in this case, the bar forms the fulcrum, the coals are the weight, the pressure of the hand on the poker is the power.

When by means of a lever we are enabled to make a certain force do an amount of work, which it could not do without the intervention of the lever, we are said to *gain a mechanical advantage.* It will be seen that mechanical advantage is not necessarily gained; thus, if in the preceding example the distance between the fulcrum and the point of contact between the lever and the mass A be greater than that between the fulcrum and the point of application of the force D, mechanical advantage will be *lost;* in other words, it would be easier to move the mass A by the direct application of the force than through the intervention of such a lever.

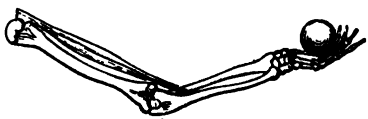
(2) In the case of the lever of the *second* kind mechanical advantage is always gained. For the weight being between the power and the fulcrum, the arm of the power is necessarily greater than that of the weight, and therefore the power less than the weight.

We have an example of such a lever in the common nutcrackers; the pressure of the hand on the extremities of the long handles of the nutcrackers supplies the power, and the resistance of the nut the force which corresponds to the weight. Another example is that of the oar of a boat; the water forms a fulcrum (though an imperfect one) for the extremity of the oar, the hand at the other extremity supplies the power, and the result is a force, greater than the power, at the rowlock, which is effective in moving the boat.

(3) In the case of the lever of the *third* kind mechanical advantage is never gained. For the arm at which the power

acts is shorter than that at which the weight acts, and therefore the power must be greater than the weight. Hence it might be imagined that this species of lever could never be advantageously applied in practice, and of course if the gaining of power be the end to be attained it never can; there is however a most interesting case of the application of this kind of lever, in which the loss of mechanical advantage is far more than compensated by the gain of advantages of another kind. The case alluded to is that of the limbs of animals, or more particularly that of the human arm.

The figure represents the skeleton of the human arm; suppose the elbow *A* to be kept at rest, and the hand to exert a force either in lifting a weight, or in pulling, or pushing; then the tendency of the hand is to revolve about *A*, and *A* will be the fulcrum, while the force exerted by the hand will correspond to the weight. Where and how will the power be applied? The power is applied near the elbow by means of certain tendons or sinews, which are acted upon by the contraction of muscles situated in the higher part of the arm. Thus the point of application of the power is between the fulcrum and the weight, and the power acts at a mechanical disadvantage; but it will be easily seen from the nature of the case, that no other kind of lever could have been conveniently adopted, because the hand must of necessity be placed at the extremity of the limb; moreover, neatness of construction and agility of motion are incomparably more important in animal mechanism than the multiplication of strength, especially in the case of man, whose natural strength must at best be small, and whose intellectual resources supply him with the means of increasing his power to an almost unlimited extent; the science of comparative anatomy, however, brings before us some curious instances of the power of the inferior animals being increased by advantageous mechanical arrangements.



A

21. We will now briefly recapitulate the results at which we have arrived in the case of the three levers respectively.

Lever of the first kind. Mechanical advantage may be either lost or gained.

Lever of the second kind. Mechanical advantage is always gained.

Lever of the third kind. Mechanical advantage is never gained.

And in all cases the principle of equilibrium is this, that the moment of the power about the fulcrum must be equal to the moment of the weight.

22. There are many other questions connected with the lever, or more generally with the theory of the *moments of forces*, which might be introduced in this place; we prefer, however, to reserve these until after we have treated of the Centre of Gravity, and have shewn how the principles already established by experiment may be placed upon a demonstrative basis.

EXAMINATION UPON CHAPTER III.

1. DEFINE the *moment* of a force about a given point. A weight of 6 lbs. is suspended from one extremity of a horizontal rod; find the weight, which suspended from the middle point would produce the same moment about the other extremity.

2. Define a *lever;* and distinguish the different kinds of lever, giving examples of each.

3. Enunciate the condition of equilibrium of a straight horizontal lever, when a weight is suspended from each extremity; and explain how the condition may be investigated experimentally.

4. Enunciate in its most general form the principle of moments, as applied to the straight lever under the action of forces perpendicular to its length.

5. Shew how the case of forces acting obliquely upon a lever may be deduced from that of forces acting at right angles to the lever.

6. In what sense can a *moment* be properly spoken of as the *product of a force and a line?*

7. Two weights of 3 lbs. and 7 lbs. respectively, hang from the extremities of a lever 1 yard long; find the fulcrum.

8. A straight rod, 6 feet long, capable of moving in a vertical plane about one extremity has a weight of 10 lbs. suspended

from its free extremity; find at what point an upward force of 35 lbs. must be applied so as to hold the rod in a horizontal position.

9. In the preceding example what will be the pressure upon the fixed extremity?

10. What is meant by *mechanical advantage being lost or gained* by the intervention of a lever? Explain under what circumstances either the one result or the other takes place in the case of each kind of lever.

11. The longer arm of a lever of the first kind is 3 feet, and the shorter 7 inches; what force will be necessary to raise a weight of a ton?

12. How much would the force in the preceding example be increased by removing the fulcrum through 1 inch towards the weight?

13. Two weights, W and W', are suspended by strings from the extremities of a lever of length a; find the fulcrum.

14. If in the preceding example a weight w be added to W, what weight must be added to W' to maintain equilibrium?

15. A straight rod, moveable in a vertical plain about a hinge at one extremity, is supported in a horizontal position by a vertical thread which is attached to it at a distance of 10 inches from the hinge; and the length of the rod is 27 inches. Supposing that the thread will support a weight of 4 oz. without breaking, find what weight may be suspended from the free extremity of the rod.

16. Two known weights, of P and Q lbs. respectively, balance upon a straight lever of the first kind; if p lbs. be added to P, the fulcrum must be shifted through a space a towards the extremity from which P hangs, in order to preserve equilibrium; and if q lbs. be added to Q, the fulcrum must be shifted through a space b towards the opposite extremity; find the length of the lever.

CHAPTER IV.

ON THE CENTRE OF GRAVITY.

1. IN every material body, or system of particles rigidly connected, there is a point which has this remarkable property, that if it be supported or fixed the body will remain at rest, whatever be the position of the body subject to the condition of that point being fixed. This point is called the *centre of gravity* of the body.

We shall be engaged in this chapter in proving the existence of the centre of gravity, in discussing some of its properties, and in determining its position in certain cases.

2. Let us take the simplest case, namely, that of two equal particles rigidly connected by a rod supposed to have no weight. Then it is evident that the middle point of the rod will be the centre of gravity; for the perpendiculars from this point upon two vertical lines drawn through the two particles will be equal, whatever be the position of the rod; therefore the moments of the weights of the particles will be equal, or the particles will be at rest.

3. Even if there be no rod joining the two particles, the middle point of the straight line joining them would be called their centre of gravity; for *if* this point were connected with the two particles, and the point were supported, the two particles *would* remain at rest in any position.

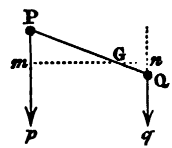
And generally, we may observe, that the centre of gravity of a body need not be a point within the body; but it may be, and frequently is, a point such that *if we conceive* the body to be rigidly connected with it the definition of the centre of gravity *would* be satisfied. For example, the centre of gravity of a hollow sphere is the centre of the sphere; for although that point has no physical connexion

with the material sphere, yet if the centre be conceived as rigidly connected with the sphere (by a rod without weight, for instance, coinciding with a diameter) it is evident that when the centre is supported the sphere will remain at rest in whatever position we place it; for the sphere being *symmetrical*, that is, of precisely similar size and shape, around the centre, when it is placed in one position there is no reason why it should change that position for another. Hence the centre of the sphere is called its centre of gravity, although there is no physical connexion between that point and the sphere itself. And so in other instances.

4. We may also observe, that a system of particles not rigidly connected are frequently spoken of as having a centre of gravity, as in the case of the two particles already discussed in Art. 3. By the centre of gravity in these cases we mean a point, which, *if it were* rigidly connected with each of the particles, *would satisfy* the definition given in Art. 1. In this sense we may speak of the centre of gravity of a pile of cannon-balls, of a quantity of water, of a piece of string.

5. Let us now consider what will be the position of the centre of gravity of two unequal particles.

Let P and Q be the two particles, and let their weights be p lbs. and q lbs. respectively; draw a straight line from P to Q, and divide it in G in such manner that



$$PG : QG :: q : p;$$

then, if from G we draw Gm, Gn perpendicular to the vertical lines Pp, Qq respectively, we shall have by similar triangles

$$PG : QG :: Gm : Gn,$$

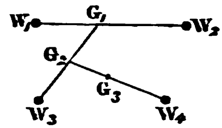
$$\text{and } \therefore Gm : Gn :: q : p.$$

Hence the *moments* of the two weights p and q about G will be equal, and therefore the two particles P and Q will balance about G; i.e. G will be the centre of gravity of P and Q.

6. We are now in a condition to prove the following general Theorem.

PROP. ***Every system of particles and every material body has a centre of gravity.***

Let W_1, W_2, W_3, W_4,... be a system of particles, the weights of which are W_1, W_2, W_3, W_4, ... respectively: suppose W_1 and W_2 joined by a rigid rod without weight, and divide this rod in G_1, so that



$$W_1G_1 : W_2G_1 :: W_2 : W_1;$$

then, from what has gone before, G_1 will be the centre of gravity of W_1 and W_2; that is, if G_1 be supported, W_1 and W_2 will balance in all positions about it, and the pressure upon the point of support will be $W_1 + W_2$.

Again, suppose G_1 and W_3 joined by a rigid rod without weight, and divide it in G_2, so that

$$G_1G_2 : W_3G_2 :: W_3 : W_1 + W_2;$$

then, if we suppose the rod W_1W_2 to rest upon the rod G_1W_3, and G_2 to be supported, the pressure $W_1 + W_2$ at G_1 and W_3 at W_3 will balance about G_2. Hence the three bodies W_1, W_2, W_3, supposed rigidly connected, will balance in all positions about G_2.

Similarly, we may find a point G_3 in the line joining G_2 and W_4, about which W_1, W_2, W_3, W_4 will balance in all positions; and so of any number of particles. Hence every system of particles has a centre of gravity.

And this proposition includes the case of all material bodies, since a body may always be conceived to be made up of an indefinite number of component particles.

Hence, every system, &c. Q.E.D.

7. If the centre of gravity of a system be supported, it is evident that the pressure upon the support will be precisely the same as if the whole system were compressed into a single particle having for its weight the sum of the weights of the particles of the system. This is sometimes expressed by saying, that *the statical effect of a system of particles is the same as if the system were collected at its centre of gravity.*

8. PROP. *Every material system has only one centre of gravity.*

For, suppose there are two, and let the system be so turned that the two centres of gravity lie in the same

horizontal plane. Then the weights of the different particles of the system form a system of vertical forces, which must have a vertical resultant passing through each of the centres of gravity; otherwise the system could not balance about each of those points; hence the vertical resultant must pass through two points in the same horizontal plane, which is absurd. Hence every material system, &c. Q.E.D.

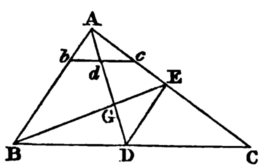
9. We shall now proceed to find the position of the centre of gravity in a few actual cases. The general determination of the position of the centre of gravity of a body of any given form and magnitude we shall not be able to solve, but there are a few instances in which the problem presents no difficulty.

10. *To find the centre of gravity of a physical right line, or of a uniform thin rod.*

The middle point will be the centre of gravity; for we may suppose the line or rod to be divided into pairs of equal weights equidistant from the middle point, and the middle point will be the centre of gravity of each pair, and therefore of the whole system, that is, of the line or rod itself.

11. *To find the centre of gravity of a plane triangle.*

Let ABC be the triangle; bisect BC in D, and join AD; draw any straight line bdc parallel to BC, and meeting AD in d; then by similar triangles, we have



$$bd : BD :: Ad : AD$$
$$:: cd : CD,$$
$$\text{or } bd : cd :: BD : CD;$$

but BC is bisected in D, therefore bc is bisected in d. Hence the line bc will balance about the point d in all positions; similarly, all lines in the triangle parallel to BC will balance about points in AD, and therefore the centre of gravity of the whole triangle must lie in AD.

In like manner, if we bisect AC in E, and join BE, the centre of gravity must be in BE; hence G, the intersection of AD and BE, is the centre of gravity of the triangle ABC.

Join DE, which will be parallel to AB. (EUCLID, VI. 2.)

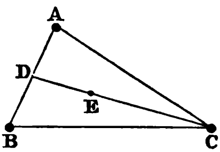
Then the triangles ABG, DEG are similar;

$$\therefore AG : GD :: AB : DE$$
$$:: BC : DC$$
$$:: 2 : 1,$$
$$\text{or } AG = 2\,GD,$$
$$\text{and } \therefore AD = 3\,GD.$$

Hence, if we join an angle of a triangle with the bisection of the opposite side, the point which is two-thirds of the distance down this line from the angular point is the centre of gravity of the triangle.

12. *To find the centre of gravity of three equal bodies placed so as to form a triangle.*

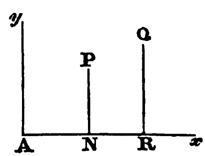
Let A, B, C be the three bodies; join AB, BC, CA. Bisect AB in D, then D will be the centre of gravity of A and B, and we may suppose A and B to be collected at D. (Art. 7.) Join CD, and take DE equal to one-third of CD; then $CE = 2\,DE$, and therefore if we consider CD as a lever with fulcrum E, the two bodies A and B suspended from D will balance the body C suspended from C, and therefore E is the centre of gravity of the three bodies.



Cor. From this it appears that the centre of gravity of a plane triangle is the same as that of three equal bodies placed at its angular points.

13. We shall now shew how to find the centre of gravity of any number of particles in the same plane; but before doing so, we must shortly explain how the position of any number of particles may be most conveniently represented mathematically.

Let P be any point, the position of which we wish to describe: take any point A, and through it draw two straight lines, Ax, Ay, at right angles to each other; from P draw PN perpendicular to one of these lines, as Ax: then it will be easily seen, that if the length of AN be given, and also the length of PN, the position of P will be entirely described. In like manner the position of any other point Q may be

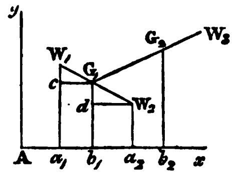


determined. We may remark that this mode of assigning the positions of points is very general in modern mathematics, and that it is usual to call AN, PN the *co-ordinates* of the point P, and to call Ax and Ay the *axes of co-ordinates.*

This being premised, let it be required

14. *To find the centre of gravity of any number of particles which lie in the same plane.*

Let W_1 W_2 W_3... be the weights of the particles; in the plane in which they lie take any two straight lines Ax, Ay at right angles to each other, as axes of co-ordinates. Draw W_1a_1, W_2a_2,...perpendicular to Ax; and let $Aa_1=x_1$, $W_1a_1=y_1$, $Aa_2=x_2$, $W_2a_2=y_2$, &c., also let x, y be the co-ordinates of the centre of gravity of the system; then it is evident that if we find x and y, we shall have solved the problem.



Join W_1 W_2, and let G_1 be the centre of gravity of W_1, W_2; from G_1 draw G_1b_1 perpendicular to Ax, and G_1c perpendicular to W_1a_1, also from W_2 draw W_2d perpendicular to G_1b_1; then, by the fundamental property of the centre of gravity, we have

$$W_1 \times W_1G_1 = W_2 \times W_2G_1;$$

but since the triangles W_1G_1c, G_1W_2d are similar, we have

$$\begin{aligned} W_1G_1 : W_2G_1 &:: G_1c : W_2d, \\ &:: a_1b_1 : a_2b_1, \\ &:: Ab_1 - x_1 : x_2 - Ab_1; \end{aligned}$$

$$\therefore\ W_1(Ab_1 - x_1) = W_2(x_2 - Ab_1),$$

$$\text{or } Ab_1 = \frac{W_1x_1 + W_2x_2}{W_1 + W_2}.$$

If we consider another particle W_3, we may, in searching for the centre of gravity of the three W_1, W_2, W_3, suppose the two former to be collected at their centre of gravity G_1; hence if G_2 be the centre of gravity of the three particles, and we draw G_2b_2 perpendicular to Ax, we have

$$Ab_2 = \frac{(W_1 + W_2)\,Ab_1 + W_3x_3}{(W_1 + W_2) + W_3};$$

and if we put for Ab_1 its value already found, we have

$$Ab_2 = \frac{W_1x_1 + W_2x_2 + W_3x_3}{W_1 + W_2 + W_3};$$

and so on for any number of particles. Hence, we shall have

$$x = \frac{W_1x_1 + W_2x_2 + W_3x_3 + \dots}{W_1 + W_2 + W_3 + \dots}.$$

And in exactly the same manner we should find that

$$y = \frac{W_1y_1 + W_2y_2 + W_3y_3 + \dots}{W_1 + W_2 + W_3 + \dots}.$$

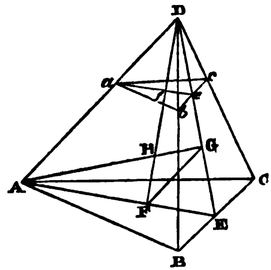
It will be seen that these formulæ express this truth, that *the centre of gravity of a system of particles is such, that the moment about any point A of the sum of their weights collected at the centre of gravity is equal to the sum of the moments of the weights.*

15. The preceding investigations refer to systems of particles lying all in one plane, or to plane bodies. It may be as well to remark concerning such bodies, that physically they can have no existence, that is to say, it is impossible that we can have a body which has length and breadth but no thickness. When we spoke of the centre of gravity of a plane triangle, it would have been more correct to speak of the centre of gravity of a portion of matter bounded by two plane triangles the surfaces of which are parallel, and very near to each other, or of the centre of gravity of a very thin frustum of a prism on a triangular base. No confusion however can arise from speaking of the centre of gravity of a plane figure, if the student bears in mind that his results are applicable to indefinitely thin plates or laminæ, or that if the thickness be considered he must regard the centre of gravity as lying inside the body and at equal distances from the two bounding surfaces.

There are few cases in which we can find the centre of gravity of solids, without more powerful mathematical appliances: one or two cases however are within our reach.

16. *To find the centre of gravity of a pyramid on a triangular base.*

Let $ABCD$ be the pyramid. Bisect BC in E; join



AE: take EF equal to $\frac{1}{3}AE$, and join DF.

Suppose the pyramid to be made up of thin triangular slices parallel to ABC, and let abc be one of them; let afe be the line in which it is intersected by the plane DAE, e and f lying in bc and DF respectively.

Then by similar triangles,

$$be : eD :: BE : ED,$$
$$\text{also } ce : eD :: CE : ED;$$
$$\therefore be : ce :: BE : CE,$$
$$\text{but } BE = CE; \therefore be = ce.$$

In like manner it may be shewn that

$$fe : af :: FE : AF,$$
$$\text{but } AF = 2FE; \therefore af = 2fe.$$

Hence f is the centre of gravity of the triangular slice abc. Similarly it will appear that the centres of gravity of all slices of the pyramid made by planes parallel to ABC lie in DF, and therefore the centre of gravity is in that line.

Similarly, if we join DE, take $GE = \frac{1}{3} DE$, and join AG, the centre of gravity will be in AG; therefore H, the intersection of DF and AG, is the centre of gravity of the pyramid.

Now join GF; then, by similar triangles,

$$HF : HD :: GF : AD,$$
$$:: FE : AE,$$
$$:: 1 : 3;$$
$$\therefore HF = \tfrac{1}{3}HD = \tfrac{1}{4}DF.$$

Hence, if we join the vertex of the pyramid with the centre of gravity of the base, and set off one-fourth of this line from the latter point, we shall determine the centre of gravity of the pyramid.

17. It is not difficult to see that the same construction will hold for a pyramid upon any base. That is to say, if

we join the vertex with the centre of gravity of the base, and set off one-fourth of this line from the latter point, we shall determine the centre of gravity of the pyramid. The student may prove for himself that this is so.

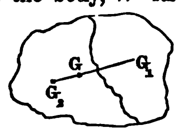
18. And still further the same construction will hold for a cone, either right or oblique; for the base may be regarded as a polygon having an infinite number of sides, and the cone as a particular case of the pyramid.

19. It was shewn in Art. 12, how we may find the centre of gravity of three equal bodies placed so as to form a triangle; and in like manner we may find the centre of gravity of four equal bodies placed at the angular points of a tetrahedron, or pyramid. And as it was shewn that the centre of gravity of the three bodies in the former case coincides with the centre of gravity of the triangle, so it will be found that in the latter case the centre of gravity of the four bodies coincides with the centre of gravity of the pyramid.

20. If we have two bodies, the centre of gravity of each of which is known, we can find the centre of gravity of the two, by considering each to be condensed into its centre of gravity, and then constructing for the centre of gravity of the two as we did for that of W_1 and W_2 in Art. 14. And the same remark applies to any number of bodies.

21. Also, when the centre of gravity of a heavy body is given, and also that of any portion of it, we can find the centre of gravity of the remainder.

For let G be the centre of gravity of the body, W its weight: G_1 the centre of gravity of the given portion, W_1 its weight. Join G_1G, and in that line produced, take G_2, such that



$$G_2G : G_1G :: W_1 : W - W_1.$$

Then G_2 will be the centre of gravity required.

22. The following general proposition concerning the centre of gravity, contains the property which is most important in a practical point of view.

Prop. *When a body is placed upon a horizontal plane, it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.*

Suppose the vertical line GC through the centre of gravity G to fall within the base, as in fig. I: then we may suppose the whole weight of the body to be a vertical pressure W acting in the line GC; this will be met by an equal and opposite pressure W from the plane on which the body is placed, and so equilibrium will be produced and the body will stand.

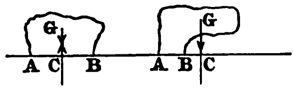


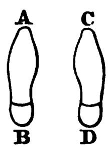
Fig. I. Fig. II.

But suppose, as in fig. II, that the line GC falls without the base; then there is no pressure equal and opposite to W at C, and therefore W will produce a moment about B, (the nearest point in the base to C,) which will make the body twist about that point and fall.

23. Hence, we see that it is not necessary that the walls of a lofty building should be accurately vertical; this is, in fact, a condition which is very often not satisfied. There are some very remarkable deviations from verticality; the leaning tower of Pisa for example.

24. We have used the term *base* in the preceding proposition, to express the portion of the body which is in contact with the horizontal plane; if the body stand upon three or more points, then, by joining these points, we shall form a triangle or polygon as the case may be, and this will be the space within which the vertical from the centre of gravity must fall.

And this remark is applicable to the case of the human body. Let AB, CD be the soles of a man's shoes; join AC, BD; then the vertical line through the man's centre of gravity must fall somewhere within the space $ABDC$. This space may be enlarged by separating the feet, and the man's steadiness is correspondingly increased. If a person raise one foot from the ground, then his *base* is reduced to the sole of the other foot, and cannot be increased; his steadiness therefore is much diminished, and if he should lose his balance, he must either put the other foot down, or change his position by a *hop*, so as to bring the sole of his foot again below his centre of gravity.



Men, and indeed all animals, acquire the habit of

instinctively shifting their position so as to satisfy the condition of equilibrium; thus, if a man walking upon a narrow plank feels himself in danger of falling upon one side, he throws out the opposite arm; a woman nursing a child leans backward; a man carrying a burden upon his back leans forward; in walking up a hill we lean forward; in walking down a hill we lean backward; and so on. In like manner, a person rising from a chair must either press the body forward to bring the centre of gravity over the feet, or else must put the feet backward under the chair to produce the same effect.

One of the best illustrations of the management of the centre of gravity is that afforded by the tight-rope dancer. The tight-rope dancer carries in his hand a heavy pole, and the centre of gravity, the position of which determines whether he will fall or not, is that of the pole and himself, regarded as two bodies. Hence the dancer to a certain extent carries his centre of gravity in his own hands, and can shift its position, so as to keep it within the narrow limits required, with much greater ease than he could if unassisted by the pole.

25. It would seem from what has been proved, that a body would rest on a horizontal plane, when supported by a single point, provided that it be so placed that the centre of gravity is in the vertical line passing through that point, which in this case forms the base. And in fact a body so situated would be, mathematically speaking, in a position of equilibrium, though practically the equilibrium would not subsist; this kind of equilibrium and that which is practically possible are distinguished by the names of *unstable* and *stable*. Thus an egg will rest upon its side in a position of *stable* equilibrium, but the position of equilibrium corresponding to the vertical position of its axis is *unstable*. So likewise there is a mathematical position of equilibrium for a needle resting on its point, or a pyramid or cone upon its apex, though such positions are obviously unstable.

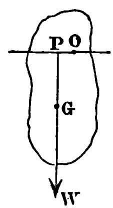
The distinction between *stable* and *unstable* equilibrium may be enunciated generally thus: Suppose a body or a system of particles to be in equilibrium under the action of any forces; let the system be arbitrarily displaced very slightly from the position of equilibrium; then if the forces be such that they tend to bring the system back to its position of equilibrium, the position is *stable*, but if they tend to move

the system still further from the position of equilibrium it is *unstable*.

26. The following property of the centre of gravity is nearly analogous to that of Art. 22.

Prop. *When a heavy body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.*

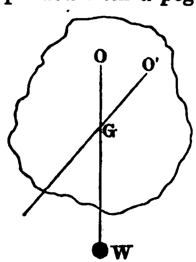
For let O be the point of suspension, G the centre of gravity, and suppose that G is not in the vertical line through O; draw OP perpendicular to the vertical through G, that is, to the direction in which the weight of the body W acts. Then the force W will produce a moment $W.OP$ about O as a fulcrum, and there being nothing to counteract the effect of this moment, equilibrium cannot subsist.



Hence G must be in the vertical line through O, in which case the weight W produces only a pressure on the point O, which is supposed immoveable.

27. From the property proved in the preceding Article it is easy to deduce a method of determining practically the position of the centre of gravity of any heavy body bounded by two parallel planes.

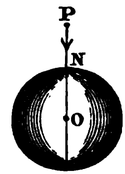
For this purpose let the body be suspended from a peg at any point O, in such a manner that it can easily turn about O; then by Art. 26 the centre of gravity will be somewhere in the vertical line through O. If, therefore, from O we suspend a plumbline, that is, a fine thread carrying at its extremity the weight W, and draw a fine line upon the bounding surface to mark the line of contact of the surface and the plumbline, the centre of gravity will be somewhere in this line. Again, let the body be suspended from a peg inserted at any other point O', and let a second line be traced by means of the plumbline, as before described. Then the point of intersection (G) of the two lines which have been traced will be the centre of gravity of the body.



28. Before leaving the subject of the Centre of Gravity, it may be well to make a few remarks upon the subject

of Gravity itself. One of the most general and most wonderful, though most simple, of the laws, to which modern science has conducted us, is this, that every particle of matter attracts every other particle of matter towards itself according to a regular law. This law, which is called that of *gravity* or *gravitation*, must be assumed to be true; the student at present taking the assertion upon trust, that the evidence in favour of the truth of the law is quite irresistible to all minds capable of following the steps of the proof.

In consequence of this law of gravitation every particle of the Earth's mass attracts every particle of a body at its surface; and if we suppose (which is very nearly but not quite true) that the Earth is a *sphere*, then the *resultant* of the attractions of the particles of which it is composed upon a particle at the outside of it will be a force tending towards the centre of the sphere. Let, for instance, O be the centre of the Earth, P a particle anywhere outside the Earth; then every particle of the Earth's mass tends to draw P towards itself; and since these particles are all symmetrically arranged round the line OP, it is evident that the resultant of all their attractions must be a force in the direction PO. And *this resultant force is that which constitutes the weight of the particle* P.



Now if P were a stone which was let fall, it would of course fall in the direction of PO, or the direction of gravity, and this direction we will call the *vertical* direction; and the plane perpendicular to PO, at the point N, where the stone strikes the Earth's surface, we will call the *horizontal plane* at that point. It is clear that the vertical directions at two different points of the Earth's surface cannot be the same, that is, they cannot be parallel, because they meet in O; but O is at a great distance from the surface, nearly 4000 miles, and, therefore, if we take two points on the Earth's surface at no great distance from each other, the vertical directions at those two points will be *nearly parallel.* For example, take two places a quarter of a mile apart; the circular measure of the angle between the vertical directions at those points will

$$= \frac{1}{4} \times \frac{1}{4000} = \frac{1}{16000}.$$

This is a very small angle, amounting to only a few seconds; hence even at the distance of a quarter of a mile from each other the directions of gravity at two places may be taken to be parallel. In all problems, therefore, concerning heavy bodies, we treat of gravity as a *force which acts in parallel lines.*

The Earth instead of being spherical, as we have supposed, is what is called a spheroid; that is, it is slightly flattened at the poles, and if we were to take a section of it by a plane passing through its centre and its poles, it would be an ellipse of which the axes would be nearly equal*. The earth being of this form, we cannot conclude that the force of gravity must at each point tend towards its centre†; we can, however, describe very simply the exact direction of gravity at any place upon the earth's surface; it is found that *the direction of gravity is the straight line perpendicular to the surface of still water at the given place;* this is a result which may be verified by experiment with an extreme decree of precision, and which also agrees with the results of mathematical investigation. Instead, therefore, of the definitions of *vertical* and *horizontal,* which were given just now, we ought, more properly, to speak of the *vertical at any place* as the straight line perpendicular to the surface of still water; and the plane perpendicular to it, that is, the surface of the water itself as the *horizontal plane.* For all common purposes, however, we may regard bodies as tending to fall towards the Earth's centre; and even if we take the more accurate definition of the direction of gravity, our former conclusion will be true, namely, that gravity may be considered as a force which acts in parallel lines.

* The *polar* diameter is 7899 miles, the *equatorial* 7925.

† There is another slight cause of deviation not considered here, namely, the rotation of the earth about its axis. The discussion of this belongs to Dynamics, but the effect is extremely small, and may be neglected.

EXAMINATION UPON CHAPTER IV.

1. Define the centre of gravity of a body, or of a system of material particles.

2. Find the centre of gravity of two unequal particles.

3. Prove that every system of particles has one centre of gravity, and only one.

4. Explain and illustrate the statement that *the statical effect of a system of particles is the same as if the system were collected at its centre of gravity.*

5. Find the centre of gravity of a physical right line.

6. A straight wire 3 feet long is composed of two pieces of 2 feet and 1 foot respectively. The former is composed of matter which weighs 1 oz. per foot, and the second of matter which weighs $2\frac{1}{2}$ oz. per yard; find the centre of gravity of the whole wire.

7. Find the centre of gravity of a triangle.

8. Find the centre of gravity of three equal bodies placed so as to form a triangle.

9. Find the centre of gravity of any number of particles which lie in the same plane.

10. The centre of gravity of a body being given, and also that of a given portion of it, shew how to find that of the remainder.

11. An equilateral triangle is divided into two parts by a straight line which bisects two of the sides; find the centre of gravity of the quadrilateral portion.

12. When a body is placed upon a horizontal plane, it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.

13. Distinguish between *stable* and *unstable* equilibrium.

14. When a heavy body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.

15. Find the centre of gravity of a pyramid on a triangular base.

16. Shew how to find the centre of gravity of a quadrilateral figure.

17. A body cannot be in stable equilibrium upon a horizontal plane if it rests on less than three supports, the supports being supposed to terminate in points.

18. Two unequal physical lines cross each other, and are attached at the point of their intersection: find their centre of gravity.

19. Find the locus of the centres of gravity of all right-angled triangles which can be described upon a given base.

20. If the sides of a triangle ABC be bisected in the points D, E, F; then the centre of the circle inscribed in the triangle DEF will be the centre of gravity of the perimeter of ABC.

21. A given number of weights (n), which are in geometrical progression, are placed at equal distances along a straight line: find their centre of gravity.

22. How may the centre of gravity of a plane figure be found experimentally?

23. Of all triangles upon the same base and having the same vertical angle, the isosceles is that of which the centre of gravity is nearest to the base.

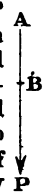
24. Two rods of the same thickness, one of which is twice as long as the other, are attached by two of their extremities so as to be at right angles to each other. Find at what angle either of them will be inclined to the vertical, when they are suspended by a string or tack at the right angle.

CHAPTER V.

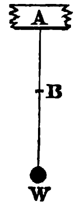
DEMONSTRATIVE MECHANICS. PARALLELOGRAM OF FORCES. EQUILIBRIUM OF A PARTICLE.

1. IN a preceding chapter we explained fully the meaning of the proposition called the "Parallelogram of Forces," and we deduced its truth by means of experiment: we are now about to shew how the truth of the same proposition may be demonstrated, without recourse to experiment, by means of an axiom concerning force. We have pursued this course, not because it is necessary, but because it appears fitted to help the student over those difficulties which belong to the first study of the Science of Mechanics. The student has (we presume) made himself acquainted with Algebra, Geometry, and Trigonometry, before he enters upon Mechanics; but those subjects are entirely confined to the properties of *space* and *number*, and he is likely therefore to feel considerable difficulty if he is thrown at once upon the demonstration of propositions concerning that which is so new to him in its character and properties as *force*. Now it is hoped that by the study of the preceding chapters this difficulty will be obviated, and that being now thoroughly familiar with the propositions which he has to prove, he will not find any very great obstacle in the way of comprehending the proof.

2. The principle upon which we shall found the proof of the Parallelogram of Forces is this: *a force acting upon a particle may be supposed to act at any point in the line of its direction, that point being conceived to be rigidly connected with the particle.*



Thus let A be a particle, acted upon by a force P in the direction AP; take B any point in AP, then we may suppose P to act at B instead of A, provided A and B be conceived to be rigidly connected together. This is a principle the truth of which will be easily seen; it does not require any experiment to prove it, but may be regarded as an axiom, the truth of which the student cannot fail to see if he has really understood what is meant by force. The principle may however be illustrated thus. Suppose W to be a weight hanging from a fixed point A by a fine string, the weight of which may be neglected; then there will be a certain pressure at A which will be equal to W: again, let us put a tack through the string at any point B, so that the weight will hang from B instead of A, then the pressure on B will be equal to W, and therefore the same as it was at A in the former case: hence if we regard W as a force producing a pressure in the line ABW, we may say that W may be supposed to act either at A or at B.



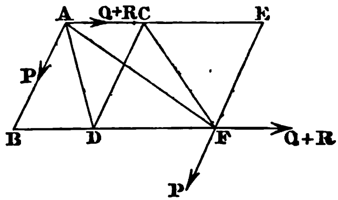
This being premised, we shall proceed to give a demonstration of the Parallelogram of Forces, which has been already enunciated in p. 10; we shall find it convenient to divide the proposition into two, in the first of which we shall consider the *direction* of the resultant of two forces, and in the second its *magnitude.*

3. Prop. *If two forces, acting on a particle at* A, *be represented in direction and magnitude by the straight lines* AB, AC, *then the resultant will be represented in* direction *by the diagonal* AD *of the parallelogram described upon* AB, AC.

(1) When the forces are *equal,* it is manifest that the direction of the resultant will *bisect* the angle between the directions of the forces: or, if we represent the forces in direction and magnitude by two straight lines drawn from the point at which they act, the diagonal of the parallelogram described upon these lines will be the direction of the resultant. Hence the proposition is true for *equal* forces.

(2) Next, suppose that the proposition, just proved for equal forces, is true for two unequal forces P and Q, and also for P and R: we shall shew that it will be true for P and $Q + R$.

Let A be the point of application of the forces; take AB to represent P in direction and magnitude, and AC to represent Q; complete the parallelogram $ABDC$, then *by hypothesis* AD is the *direction* of the resultant of P and Q; and since a force may be supposed to act at any point of its direction, we may consider D as the point of application of the resultant of P and Q; therefore, since the resultant is in all respects equivalent to its components, we may suppose the forces P and Q themselves to act at D, P parallel to AB, and Q parallel to AC; or still further we may suppose P to act at C, in the direction CD.



Again; the force R which acts at A may be supposed to act at C; take CE to represent it in direction and magnitude, and complete the parallelogram $CDFE$; then *by hypothesis*, CF is the direction of the resultant of P and R acting at C: hence the resultant of P and R may be supposed to act at F, or P and R may be supposed themselves to act at that point parallel to their original directions.

Lastly; the force Q, which at present is supposed to be acting at D in the direction DF, may be supposed to act at F.

Hence we have reduced the forces P and $Q + R$, acting at A, to P and $Q + R$, acting at F; consequently F is a point in the line of action of the resultant, and therefore AF is the direction of the resultant: that is, if the proposition be true for P and Q, and also for P and R, it is true for P and $Q + R$.

But the proposition is true for P and P, and also for P and P, therefore it is true for P and $P + P$ or $2P$; therefore for P and $P + 2P$ or $3P$; and so on; therefore generally for P and mP, where m is any whole number.

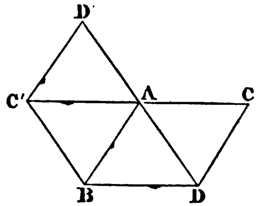
In like manner the proposition may be extended to mP and nP, m and n being any whole numbers. We may therefore consider the proposition to be true for all forces*.

∴ If two forces, &c. Q.E.D.

* Rather for all *commensurable* forces; that is, for all forces the ratio of whose magnitudes can be expressed by the ratio of two whole numbers. But this is not the case with all forces; for instance, we might have two forces,

4. **Prop.** *If two forces, acting on a particle at* A, *be represented in direction and magnitude by the straight lines* AB, AC, *then the resultant will be represented, not only in direction, but also in* magnitude, *by the diagonal* AD *of the parallelogram described upon* AB, AC.

Produce the diagonal DA to D', making AD' *equal* to the resultant of AB in AC in *magnitude;* complete the parallelogram $ABC'D$, and join AC'.



Then since AD' is *equal* to the resultant of AB, AC, and drawn in the direction opposite to that of the resultant, the three forces AB, AC, AD' balance each other, and therefore any one of them is in the direction of the resultant of the other two; hence AC is in the direction of the resultant of AB, AD'; but AC' is also in that direction, therefore AC, AC' are in the same straight line. Hence $ADBC'$ is a paral-

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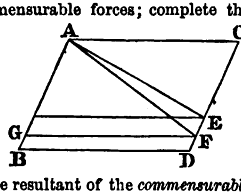
one measured by $\sqrt{3}$ lbs. and the other by 2 lbs.; now the ratio of $\sqrt{3} : 2$ cannot be expressed by the ratio of two whole numbers *exactly*, though it can be so expressed *as nearly as ever we please.* To make this more clear, observe that $\frac{\sqrt{3}}{2} = \cdot 8660254$ very nearly; hence $\sqrt{3} : 2 :: 8660254 : 10000000$, very nearly, and by taking more decimal places we could make the approximation still more close. Now as the proposition proved in the text is true for two forces whose ratio is 8660254 : 10000000, we should seem to be safe in concluding that it is also true for the *incommensurable* forces whose ratio is $\sqrt{3} : 2$. And by considering the matter thus we might conclude, that the proposition proved in the text for *commensurable* forces is true also for *incommensurable.* To take away however all kind of doubt we subjoin the following *reductio ad absurdum.*

Let AB, AC represent any two incommensurable forces; complete the parallellogram $ABDC$, and if AD be not the direction of the resultant, let it be AE. Suppose AC to be divided into a number of equal parts, each part being less than ED, and suppose distances of the same magnitude to be set off along CD, beginning at C, then one of the divisions must fall between E and D; let F be the point which marks the division, and complete the parallelogram $AGFC$, then AF is the direction of the resultant of the *commensurable* forces AG, AC: but AF makes a larger angle with AC than AE, that is, the resultant of AG and AC lies further away from AC than the resultant of AB and AC, although AG is less than AB, which is absurd: hence AE is not the direction of the resultant; and it may be shewn in like manner that no line is in that direction except AD. Hence the proposition proved in the text for *commensurable* forces, is true also for *incommensurable.*



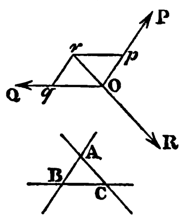
lelogram; therefore $AD = BC'$: but $BC' = AD'$: therefore $AD = AD'$. And by construction AD' represents the resultant of AB and AC in magnitude; therefore AD also represents the resultant.

$\therefore$ If two forces, &c. Q.E.D.

5. We have thus established the proposition known as the Parallelogram of Forces, without appeal to experiment as in Chap. II.; and the proposition may now be supposed to rest upon the same kind of evidence as the theorems of Euclid.

6. The proposition is sometimes stated in a form, in which it is called the *Triangle of Forces.* We will enunciate it as follows.

PROP. *If three forces, acting in the same plane, be in equilibrium upon a particle, and if in that plane we draw any three straight lines parallel to the directions of the forces, then the three sides of the triangle so formed will be in the same proportion as the forces.*



Let O be the particle, P, Q, R the forces; upon the directions OP, OQ set off Op, Oq proportional to P and Q: complete the parallelogram $Oprq$, and join Or, then Or is in the same straight line with OR, by the parallelogram of forces; and the three lines Op (or rq), Oq, Or are proportional to P, Q, R, respectively.

Now draw three straight lines AB, BC, AC, parallel to the directions of P, Q, R respectively, that is, parallel to rq, Oq, Or; then the triangle ABC so formed is similar to the triangle rqO;

$$\therefore AB : BC : AC :: rq : Oq : Or,$$
$$:: P : Q : R.$$

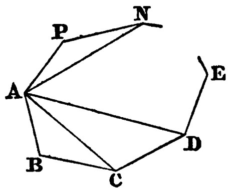
$\therefore$ If three forces, &c. Q.E.D.

COR. Hence, if two sides of a triangle, taken in order from an angular point, represent in magnitude and direction two forces which act at that point, then the third side, *not taken in the same order as the other two,* will represent the resultant. Thus if AB, BC represent two forces acting on a particle at A, then AC (*not* CA) will represent the resultant.

7. We may generalize this proposition still further, and deduce what may be called the *Polygon of Forces.*

PROP. *If the sides of a polygon* AB, BC, CD, DE,... NP, PA, *represent in magnitude and direction forces acting upon a particle, these forces will produce equilibrium; and any one of the sides, as* AP, *taken in the opposite direction to that above supposed, will represent the resultant of all the rest.*

Join AC, then AC represents the resultant of AB, BC.



Join AD, then AD represents the resultant of AC, CD, i.e. of AB, BC, CD.

And so on: hence AN represents the resultant of AB, BC, CD, DE......

But the forces represented by AN, NP, PA, are in equilibrium; hence the forces represented by AB, BC, CD, DE,...NP, PA are in equilibrium.

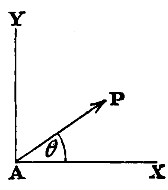
Hence the first part of the proposition is true; and the second immediately follows.

∴ If the sides of a polygon, &c. Q.E.D.

OBS. It may be remarked that the straight lines, AB, BC, &c., need not be all in the same plane.

8. The student is already acquainted with the application of the parallelogram of forces to the resolution of a force into two components in any directions (Art. 12, p. 12), and he will remember that we especially called attention to that case of resolution in which the components are at right angles to each other (Art. 14, p. 14). We shall now proceed to apply the principle of resolution and composition of forces to the following very general proposition.

PROP. *Any number of forces act at the same point, their directions all lying in the same plane; to find the direction and magnitude of the resultant.*



Let P be any one of the forces acting at the point A. Let the plane of the paper be that in which the forces act; in that plane choose any two lines at right angles to each other, AX and AY, and let θ be the angle which the direction of P makes with AX. Then P is equivalent to

$P \cos\theta$ acting in the direction AX,
together with $P \sin\theta$ .... AY.

In like manner, a force P', the direction of which makes an angle θ' with AX, is equivalent to

$P' \cos\theta'$ acting in the direction AX,

together with $P' \sin\theta'$ .... AY.

And so on for any number of forces. Hence, adding together the forces which act in the same direction, we shall have a system of forces P, P', ... acting at angles θ, θ' ...... with the line AX, equivalent to

$P\cos\theta + P'\cos\theta' + ...$ acting in the direction AX,

together with $P\sin\theta + P'\sin\theta +$ .... AY.

For shortness' sake, let

$$P\cos\theta + P'\cos\theta' + ... = X,$$

and

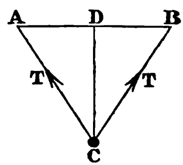
$$P\sin\theta + P'\sin\theta' + ... = Y;$$

and let R be the required resultant, ϕ the angle which its direction makes with the line AX; then

$$R\cos\phi = X,$$
$$R\sin\phi = Y;$$
$$\therefore\ \tan\phi = \frac{Y}{X},\ R^2 = X^2 + Y^2.$$

9. We subjoin some examples of the process of composition described in the preceding Article.

Ex. 1. A weight of 10 lbs. is supported by two strings, each of which is 3 feet long, the ends being attached to two points in a horizontal line 3 feet apart; to find the tension of each string.



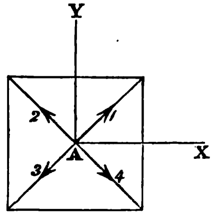
Let A, B be the two points of support; AC, BC the strings; C the weight, T the tension of either of the strings, that is the force which it exerts upon the weight in the direction of its length. Draw CD perpendicular to AB; then ABC is an equilateral triangle, and $ACD = BCD = 30^0$.

Then resolving the two tensions vertically, we have for the resolved part of each $T\cos 30^0 = T\frac{\sqrt{3}}{2}$; and the two vertical resolved parts together support the weight of 10 lbs.;

$$\therefore\ T\sqrt{3} = 10,$$
$$\text{or,}\ T = \frac{10}{\sqrt{3}}\ \text{lbs.}$$

Ex. 2. A particle placed in the centre of a square is acted

upon by forces of 1, 2, 3 and 4 lbs. respectively, tending to the angular points; to find the magnitude and direction of the resultant force.

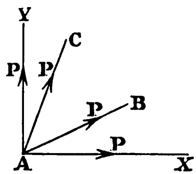


Let A be the particle, and draw two straight lines AX, AY perpendicular to the sides of the square, as in the figure. Then it will be seen that the application of the formulæ of the preceding article gives us the following;

$$R\cos\phi = 1\times\cos 45^0 + 2\times\cos 135^0 + 3\times\cos 225^0 + 4\cos 315^0,$$
$$= \cos 45^0 - 2\cos 45^0 - 3\cos 45^0 + 4\cos 45^0 = 0;$$
$$R\sin\phi = 1\times\sin 45^0 + 2\times\sin 135^0 + 3\times\sin 225^0 + 4\sin 315^0,$$
$$= \sin 45^0 + 2\sin 45^0 - 3\sin 45^0 - 4\sin 45^0 = -4\sin 45^0,$$
$$= -2\sqrt{2}.$$

$\therefore\ \phi = 90^0$, and $R = -2\sqrt{2}$; *i. e.* the resultant is a force of $2\sqrt{2}$ lbs. acting in the direction opposite to AY.

Ex. 3. If four equal forces act by strings upon a particle, the angles between the directions being 30^0, 45^0, and 15^0, to find the direction in which the particle will begin to move.



Let A be the particle; then the first and last strings are at right angles to each other; therefore we may conveniently take them as corresponding to the lines of reference which we have called AX and AY. Let AB, AC be the other two strings, and P the force exercised by each. Then we shall have

$$R\cos\phi = P + P\cos 30^0 + P\cos 75^0,$$
$$R\sin\phi = P\sin 30^0 + P\sin 75^0 + P;$$

$$\therefore\ \tan\phi = \frac{1+\sin 30+\sin 75^0}{1+\cos 30+\cos 75^0} = \frac{1+\frac{1}{2}+\frac{\sqrt{3}+1}{2\sqrt{2}}}{1+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{3\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}+\sqrt{6}+\sqrt{3}-1}.$$

This formula may be reduced to numbers, and ϕ determined by means of trigonometrical tables.

10. We have already enunciated in the form of the *polygon of forces* the most general conditions of the equilibrium of a system of forces acting on a particle; this may be

called the *geometrical* form of the conditions of equilibrium. We shall now investigate the conditions algebraically.

PROP. *To find the conditions of equilibrium of any system of forces, acting in one plane at the same point.*

Suppose the forces to be all reduced to one (R), as in Art. 8; then in order that there may be equilibrium, we must have

$$R = 0,$$
$$\text{or } X^2 + Y^2 = 0.$$

And this equation cannot be true, unless we have

$$X = 0 \text{ and } Y = 0;$$
$$\text{or } P\cos\theta + P'\cos\theta' + \ldots = 0,$$
$$P\sin\theta + P'\sin\theta' + \ldots = 0.$$

These are the conditions of equilibrium; and they may be expressed in words by saying, that *the sum of the forces resolved in any two directions perpendicular to each other must vanish.*

11. We shall now illustrate these principles of equilibrium by applying them to several examples.

Ex. 1. If three forces, P, Q, R, be in equilibrium upon a point O; then

$$P : Q : R :: \sin QOR : \sin ROP : \sin POQ.$$

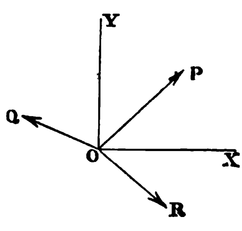
This immediately follows from the *triangle of forces.* For referring to the figure of Art. 6, we have

$P : Q : R :: AB : BC : AC,$

$:: \sin ACB : \sin BAC : \sin ABC$, by Trigonometry,

$:: \sin QOR : \sin ROP : \sin POQ.$

It will, however, be worth while to deduce the result from the principle of the preceding Article.



Draw any two straight lines OX, OY at right angles to each other, and let $XOP = \theta$, $XOQ = \phi$, $XOR = \psi$, these angles being all measured the same way round from OX.

Then, in order that P, Q, R may be in equilibrium, we must have

$$P\cos\theta + Q\cos\phi + R\cos\psi = 0,$$
$$P\sin\theta + Q\sin\phi + R\sin\psi = 0.$$

Multiply these equations by $\sin\psi$ and $\cos\psi$ respectively, and subtract; then we have

$$P(\sin\psi\cos\theta - \sin\theta\cos\psi) + Q(\sin\psi\cos\phi - \sin\phi\cos\psi) = 0,$$

or $\qquad P\sin(\psi - \theta) + Q\sin(\psi - \phi) = 0;$

but $\qquad \psi - \theta = 360^0 - ROP$, and $\psi - \phi = QOR$,

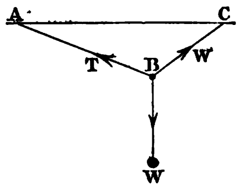
$$\therefore \ -P \sin ROP + Q \sin QOR = 0,$$

$$\text{or,} \quad \frac{P}{\sin QOR} = \frac{Q}{\sin ROP}.$$

Hence we may conclude that

$$\frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ}, \text{ as before.}$$

Ex. 2. A small ring B is attached to the extremity of a thread AB, which is fastened at A. CBW is another thread passing through the ring B and supporting a weight W. To find the position of B; A and C being in the same horizontal line.



We are to regard the ring B as a particle, kept at rest by three forces acting in the directions of the three portions of thread which meet in it. Concerning the force exerted by AB, in other words the *tension* of the thread, we know nothing, we will therefore denote it by a symbol T; the length of AB is given, call it l. With regard to the other thread, we observe that the force exerted by the upper portion of it, CB, must be equal to that exerted by the lower portion BW, in other words, the *tension* is the same throughout the same thread; therefore we shall have a force W in the direction BC, and another force W in the direction BW. The distance AC must be given, call it a. And let $BAC = \theta$, $ACB = \phi$.

Then, resolving the forces horizontally and vertically, we have the two following equations;

$$T \cos \theta - W \cos \phi = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$T \sin \theta + W \sin \phi - W = 0 \ \ldots\ldots\ldots\ldots\ldots (2).$$

But these equations involve *three* unknown quantities, T, θ, and ϕ, therefore we must have *one other* relation between them; this is supplied by the trigonometrical conditions of the triangle ABC; for we have

$$\frac{l}{a} = \frac{\sin \phi}{\sin (\theta + \phi)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now multiplying (1) by $\sin \theta$, and (2) by $\cos \theta$, and subtracting, there results

$$W (\sin \theta \cos \phi + \cos \theta \sin \phi) - W \cos \theta = 0,$$

$$\text{or,} \quad \sin (\theta + \phi) = \cos \theta,$$

$$\therefore \ \theta + \phi = 90^0 - \theta,$$

$$\text{or,} \quad \phi = 90^0 - 2\theta \ \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

$$\therefore \text{ from (3), } \frac{l}{a} = \frac{\cos 2\theta}{\sin \theta},$$

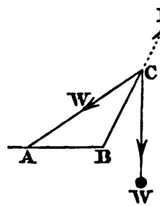
$$1 - 2 \sin^2\theta = \frac{l}{a} \sin \theta,$$

$$\sin^2\theta + \frac{l}{2a} \sin \theta + \frac{l^2}{16a^2} = \frac{l^2}{16a^2} + \tfrac{1}{2},$$

$$\sin \theta = -\frac{l}{4a} \pm \sqrt{\frac{l^2}{16a^2} + \tfrac{1}{2}}.$$

The + sign must be taken for the radical, since it is evident that θ must be less than 180^0. The value of θ thus found entirely determines the position of B.

Ex. 3. A rod BC is moveable in a vertical plane about a hinge at B; a thread, attached to a point A in the same horizontal line as B, passes over the extremity of the rod and supports a weight W. Omitting the weight of the rod, it is required to find the position of equilibrium.



In this problem we must regard the extremity C of the rod as a particle, which is kept in equilibrium by the two forces exerted by the thread in the directions CA and CW respectively, and by that which the rod itself exerts in the direction of its length. The first two forces will be each equal to W; the last we will denote by P.

Then if $AB=a$, $BC=b$, $BAC=\theta$, $ACB=\phi$, we have, by resolving the forces horizontally and vertically, the following equations,

$$W \cos \theta - P \cos (\theta + \phi) = 0 \text{ ....(1)},$$
$$W \sin \theta - P \sin (\theta + \phi) + W = 0 \text{ ....(2)};$$

and we have also the Trigonometrical condition,

$$\frac{a}{b} = \frac{\sin \phi}{\sin \theta} \text{ ....(3)}.$$

Having obtained these three equations, θ, ϕ, and P may all be found, and the problem may be completed in the same manner as the preceding one.

12. The preceding problems suggest some important remarks.

The equations with which we have to deal in Statics are of two kinds; those which arise from the two mechanical principles, namely, the Parallelogram of Forces, and the

Principle of the Lever, and those which arise from the necessary geometrical connection of the different parts of the system. In considering the equilibrium of a particle, acted upon by forces whose directions lie all in one plane, the equations which result from the Parallelogram of Forces are *two* and *only two;* and these we call the *mechanical equations* of the problem. If these equations involve only *two* unknown quantities, they contain the complete solution of the problem; but if, as is frequently the case, they contain more than two, then other relations among the unknown quantities must be sought from geometrical considerations; the equations so found, which of course contain no forces, but only lines and angles, are called *geometrical equations.* And in the solution of problems it is always necessary to obtain as many equations as there are unknown quantities involved; so that if there be n unknown quantities, we must, before we can solve the problem, obtain $n-2$ geometrical relations among them.

It may be further remarked that the greater number of statical problems may be solved in more than one way. The advantage of the general method given in p. 56 is, that it includes all kinds of problems, that it is simple in its principle and easily applicable in almost all cases. At the same time it must be allowed, that many problems may be solved more concisely by choosing methods peculiarly suitable to them. In the case of Ex. 2, p. 57, for instance, the equation (4), which together with the geometrical equation (3) contains the solution of the problem, may be obtained readily thus:

The tension T is in the direction of the resultant of the two tensions which act in the directions BC, BW. But these two latter tensions are equal; therefore AB must bisect the angle between BC and BW.

Now $WBC = 90^\circ + \phi$;

and by our principle, $\dfrac{WBC}{2} = BAC + BCA$ (Euc. I. 32).

$$\therefore\ 90^\circ + \phi = 2\theta + 2\phi,$$
$$\text{or } \phi = 90^\circ - 2\theta,$$

which is the equation required.

The same method will simplify the solution of Ex. 3. The student will often meet with cases in which a little ingenuity will save much trouble; he should however on no

account neglect the application of the uniform general method of p. 56, which, though not always the shortest, is certainly the surest. Sometimes a geometrical construction will be able to take the place of elimination amongst several equations, and, when familiar with the subject, the student may adopt in each problem the method which seems to him best; but he must remember that he is studying *Mechanics* and not *Geometry*, and therefore those methods are the most important and best, which exhibit from the clearest point of view the mechanical conditions of the problem.

EXAMINATION UPON CHAPTER V.

1. Prove the parallelogram of forces, so far as the *direction* of the resultant is concerned, for *commensurable* forces.

2. Extend the proof to the case of *incommensurable* forces.

3. Assuming the parallelogram of forces so far as the *direction* of the resultant is concerned, prove it as to *magnitude*.

4. Enunciate the *Triangle of Forces*.

5. Enunciate the *Polygon of Forces*.

6. Determine algebraically the direction and magnitude of the resultant of any number of forces acting in given directions at the same point, the directions being supposed to lie all in one plane.

7. Investigate algebraically the conditions of equilibrium of a particle under the action of any forces whose directions all lie in one plane.

8. The resultant of two forces which act at right angles to each other is equal to n times the geometrical mean between them; find the ratio of the two forces, and the smallest value of n for which the problem is possible.

9. Given the sum of two forces, and their resultant when they act at an angle of 60^0 with each other; find the forces.

10. A and B can each carry a weight of P lbs. What weight can they carry between them, when walking a feet apart, by means of two cords, each b feet long, attached to the weight?

11. Two equal weights (W) are attached to the extremities of a thread, which is suspended from three tacks in a wall, forming an equilateral triangle; find the pressure on each tack.

12. In the preceding problem find the *vertical* strain upon each tack, supposing the base of the triangle to make an angle θ with the horizon.

13. A particle at the centre of a regular hexagon is urged towards the six angular points by forces equivalent to 1, 2, 3, 4, 5, 6 lbs. respectively; determine the direction and magnitude of the resultant.

14. A, B, and C pull at the ends of three ropes which are knotted together in O: B and C are of equal strength, and A is as strong as B and C together; what help will B and C require to maintain O in equilibrium against A when $BOC=60^{\circ}$?

15. A fine thread has a small ring at one extremity; the other extremity is passed through the ring and attached to a weight; the whole is suspended by means of the loop thus formed from two smooth tacks in the same horizontal line; determine the position of equilibrium.

16. In the preceding problem find the direction and magnitude of the pressure upon the tacks.

17. A given force R is divided into two others P and Q $(P+Q=R)$; prove that the resultant of P and Q, supposed to act on a point at right angles to each other, will be least when $P=Q$.

18. Six men pull by means of a rope 100 feet long attached to the top of a tree 60 feet high towards the South; and five men by means of a rope 12 feet long towards the East; find in what direction the tree will fall.

CHAPTER VI.

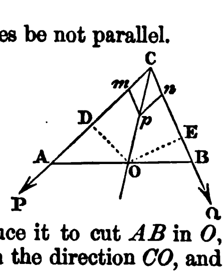
DEMONSTRATIVE MECHANICS. PRINCIPLE OF THE LEVER. THEORY OF COUPLES. CONDITIONS OF EQUILIBRIUM OF A RIGID BODY, THE DIRECTIONS OF THE FORCES BEING ALL IN ONE PLANE.

1. IN the preceding Chapter we have been concerned entirely with the equilibrium of forces acting on *a particle*, or the conditions under which *a particle* acted upon by any system of forces whose directions are in one plane *can be at rest*. In the present we shall be occupied with the conditions of equilibrium of a *rigid body;* we have already, in Chap. III., considered the particular case of two weights suspended upon a lever, and we shewed, experimentally, that the condition of equilibrium was the equality of the moments of the two weights about the fulcrum: we shall now shew how this and some more general results may be deduced from the Parallelogram of Forces, which in the preceding Chapter we have demonstrated.

We shall commence with the *Principle of the Lever.*

2. PROP. *If two forces acting at the extremities of a lever, and tending to twist it opposite ways, produce equilibrium, the moments of the forces about the fulcrum are equal.*

I. Let the directions of the forces be not parallel.



Let P and Q be the forces, acting at the extremities A, B, of the lever AB. Produce the directions of P and Q until they meet in C; then P and Q may both be supposed to act at C. Take Cm, Cn, proportional to P and Q, and complete the parallelogram $Cmpn$; join Cp and produce it to cut AB in O, then the resultant of P and Q acts in the direction CO, and

therefore O must be the fulcrum, otherwise there could not be equilibrium. Draw OD, OE perpendicular to AC, BC; then

$$\frac{P}{Q}=\frac{Cm}{Cn}=\frac{Cm}{mp}=\frac{\sin Cpm}{\sin mCp}=\frac{\sin nCp}{\sin mCp}$$

$$=\frac{OC \sin OCE}{OC \sin OCD}=\frac{OE}{OD};$$

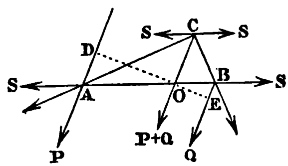
$$\therefore P.OD=Q.OE,$$

or the moments of P and Q about O are equal.

II. Let the directions of the forces be parallel.

In this case the directions of the forces, when produced, will not meet in a point, as supposed in the preceding investigation; and we must therefore slightly modify the method.

At A and B apply any two equal and opposite forces S in the direction of the lever; this will manifestly not affect the equilibrium; then the resultant of P and S will be a force in the direction CA, suppose, and that of Q and S a force in the direction CB. Suppose them both to act at C, and there to be resolved into their constituent parts P and S, Q and S; the portions S, S will destroy each other, leaving a resultant $P+Q$ in the direction CO parallel to the directions of the forces.



Then the sides of the triangle AOC are parallel to the directions of the forces P, S and their resultant; therefore by the *triangle of forces* (Art. 6, p. 52),

$$\frac{P}{S}=\frac{CO}{AO}.$$

In like manner $$\frac{Q}{S}=\frac{CO}{BO};$$

$$\therefore P.AO=Q.BO.$$

If the forces P and Q be perpendicular to the lever, this formula proves the proposition; if not, from O draw OD, OE perpendicular to the directions of P and Q; then the triangles AOD, BOE being similar, we have

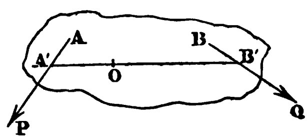
$$\frac{AO}{OD} = \frac{BO}{OE};$$

and $\therefore P.OD = Q.OE$.

Hence *If two forces*, &c. Q.E.D.

3. We have in the preceding demonstration supposed that the two forces act at the extremities of a *straight rigid rod;* but it is not difficult to see, that the proposition is true of two forces acting in the same plane upon any rigid body one point of which is fixed.

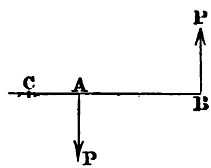
For let P and Q be two forces acting at the points A and B of a rigid body, in which the point O is fixed. Through O draw any straight line $A'OB'$, meeting the directions of P and Q in A' and B' respectively; then P may be supposed to act at A', and Q at B', and thus the problem is reduced to that of two forces acting at the extremities of the straight lever $A'OB'$.



4. We shall now proceed to the general problem of the equilibrium of any number of forces, acting in the same plane upon a rigid body. The most elegant method of treating the problem, and the simplest, is that which depends upon the properties of *couples*, which we must therefore in the first place explain.

5. Def. Two equal and opposite forces acting at right angles to a rigid rod are called *a couple.*

Let the two forces, P, P act in opposite directions upon the extremities of AB, and perpendicularly to its length, then AB is called the *arm* of the couple, and $P.AB$ is called its *moment.*

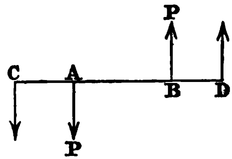


6. Now the peculiarity of a couple is this, that *it is the only case of two forces acting upon a lever, in which it is impossible to find a third force which will with the other two produce equilibrium.* If possible let C be a point in the direction of AB produced, at which a force may be applied which shall be in equilibrium with the two forces of the couple. Then by what has been already proved (Art. 2) we must have

$$P.AC = P.BC,$$
$$\text{or } AC = BC;$$

which is impossible.

The same truth may be seen from the following simple consideration. Suppose a force applied at C to be capable of keeping the system in equilibrium; then producing AB and making $BD = AC$, a force applied at D, in the direction opposite to that which we supposed applied at C,



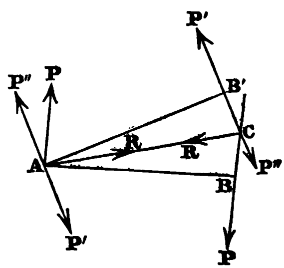
will be situated exactly in the same manner with reference to the couple as that at C: so that if a force at C can keep the system in equilibrium, an opposite force applied at D can do the same, which is absurd.

Hence we may conclude, that the effect of a couple upon a rigid rod will be to tend to make the rod *twist* about its middle point.

7. The application of the method of couples to the investigation of the conditions of equilibrium of a rigid body depends upon the three following propositions.

8. Prop. *The effect of a couple is not altered by turning its arm about one extremity through any angle in the plane of the forces.*

Let P, P be the forces, AB the arm of the couple: through A draw AB' equal to AB, and making any angle with it: at A apply two opposite forces, in the direction perpendicular to AB', and each equal to P; we shall call them P' and P'' for distinction's sake, but it will be borne in mind that they are each of the same magnitude as P. At B', in like manner,



apply the equal and opposite forces P', P'', as represented in the figure. Produce the directions of P at B, and P' at B', to meet in C; then P, P' may be supposed to act at C; join AC.

Now in the triangles BAC, $B'AC$, we have $AB=AB'$, and AC common, and the right angle ABC=the right angle $AB'C$; $\therefore BC=B'C$, and the triangles are equal in all respects.

Hence AC bisects the angle between the two equal forces P, P'; and therefore P, P' acting at C will have a resultant, (R suppose,) in the direction CA.

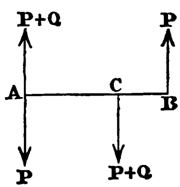
Again, since PA is parallel to PC and AC meets them, the angle $PAC=PCA$: in like manner the angle $P'AC=P'CA$; hence the forces P, P' acting at A will have a resultant R, in the direction AC.

The two forces R, R, acting at A and C, in opposite directions, will neutralize each other; and thus the only forces left are P'' at A, and P'' at B'. That is, the couple with the arm AB has been transformed into a couple with the equal arm AB', and equal forces.

$\therefore$ *The effect of a couple, &c.* Q.E.D.

9. PROP. *The effect of two couples, the arms of which have a common extremity, and which tend to twist in the same direction, is the same provided their* moments *are equal.*

Let AB be the arm of a couple; P, P the forces. At A apply a force greater than P, $P+Q$ suppose; and at C, a point between A and B, apply the force $P+Q$ in the opposite direction.



Then the two opposite forces $P+Q$ and P, acting at A, will be equivalent to a force Q acting in the direction of the former; and by what has been proved in Art. 2, the force Q at A, and the force P at B will be in equilibrium with the opposite force $P+Q$ at C, provided

$$Q \, . \, AC=P \, . \, BC,$$

$$\text{or } (P+Q)\,AC=P\,(AC+BC)=P \, . \, AB.$$

Hence the original couple will be entirely counteracted by the new couple which we have applied, which has the same moment and tends to twist in the opposite direction.

$\therefore$ *The effect of two couples, &c.* Q.E.D.

10. Taking this proposition in conjunction with the last, we see that *if one extremity of the arm be given the effect*

of a couple depends entirely upon its moment; hence it is not unusual to denote a couple by its moment; thus if we have a couple of which the forces are P, P, and the arm a, we should call it *the couple* $P.a$.

11. A couple which is equivalent to any number of couples is called the *resultant* of those couples; and those couples are called with reference to that resultant *component couples.*

12. The algebraical sign $-$, which we have found useful, as designating the direction of a force, may also be applied with advantage to couples: thus, if we have two couples, one of which tends to twist a body in one direction and the other in the opposite, we may distinguish them by the signs $+$ and $-$ attached to their moments. In the preceding proposition, for example, we obtained the result,

$$(P+Q)AC=P.AB,$$

$$\text{or } (P+Q)AC-P.AB=0.$$

If we agree to call one of these moments positive and the other negative, we shall have this result,

the algebraical sum of the moments $=0$.

This result we shall generalize and further elucidate by the following proposition.

13. Prop. *The resultant of any number of couples, the arms of which have a common extremity, is that couple which has for its moment the algebraical sum of the moments of the component couples.*

Let Pa, $P'a'$, $P''a''$,...... be the couples; and let us reduce all the couples to the arm a; thus the couple $P'a'$ will be equivalent to a couple having an arm a, and force $P'.\frac{a'}{a}$, since $P'a'=P'.\frac{a'}{a}.a$, (Art. 9); and $P''a''$ will be equivalent to a couple having an arm a, and force $P''.\frac{a''}{a}$; and so on.

Now suppose R to be the resultant of the forces acting at either end of the arm a, when the couples have been all reduced to that arm;

$$\therefore R=P+P'.\frac{a'}{a}+P''.\frac{a''}{a}+......$$

And by the process adopted, the couples are all reduced to one, having an arm a and force R;

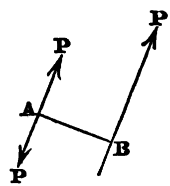
$$\therefore \text{ the moment of the resultant couple} = R \, . \, a,$$
$$= Pa + P'a' + P''a'' + \ldots\ldots$$

If the couples should not all tend to twist in the same direction, the moments of those which tend in the direction opposite to Pa will be negative.

∴ *The resultant, &c.* Q.E.D.

14. The three propositions which have been proved in Arts. 8, 9, 13, contain (as was announced) all the necessary properties of couples, when we consider the action of forces in one plane only. We shall now apply the theory of couples to the investigation of the conditions of equilibrium of a rigid body, the direction of the forces lying all in one plane.

15. PROP. *Any system of forces, (the directions of which lie in one plane,) acting upon a rigid body, may be reduced to a single force, and a single couple.*



Let P be any one of the forces, acting in the direction BP. Take any point A in the plane of the forces; and at A apply two equal and opposite forces P, parallel to BP; this will not affect the condition of the body. Draw AB perpendicular to BP. Then instead of the force P acting in direction BP we have now the force P acting at A parallel to BP, and the couple $P \, . \, AB$.

In like manner, each of the forces may be reduced to a force at A parallel to its direction, and a couple the arm of which has A for one extremity.

Now all the *forces* at A are equivalent to *one resultant force* (Art. 8, p. 53); and all the *couples* the arms of which terminate in A are equivalent to *one resultant couple* (Art. 13).

∴ *Any system, &c.* Q.E.D.

16. It is easy to see, that the resultant force spoken of in the preceding proposition will be the same wherever the point A is taken; for if P be any one of the forces, θ the angle which its direction makes with any given straight line through A, P may be resolved into $P \cos \theta$ parallel to that

line, and $P \sin\theta$ perpendicular to it (Art. 8, p. 53); and other forces P', P''... may be resolved in like manner: hence if R be the resultant force, and ϕ the angle which its direction makes with the line from which θ is measured, we have

$$R\cos\phi = P\cos\theta + P'\cos\theta' + \dots$$
$$R\sin\phi = P\sin\theta + P'\sin\theta' + \dots$$

which results are altogether independent of the position of A. But in calculating the moment of the resultant couple we must find the algebraical sum of the *moments of the forces with respect to A:* thus if $AB = a$, and the perpendicular distance from A upon the direction of P' be a', and so on, we have

$$\text{moment of resultant couple} = Pa + P'a' + \dots\dots;$$

and this quantity manifestly depends for its value upon those of a, a'...that is, upon the position of A.

17. From the preceding proposition we can at once deduce the *Conditions of equilibrium for a rigid body.* For we have already shewn that a force and a couple cannot counteract each other (Art. 6); hence, if a system of forces be reduced to one resultant force, and one resultant couple, the two must *separately vanish*, that is, we must have

$$\text{resultant force} = 0,$$
$$\text{moment of resultant couple} = 0.$$

The former of these conditions divides itself into two; for (as in Art. 10, p. 56), if a force $= 0$, each of its components must $= 0$. Hence according to the notation adopted in the preceding article, we shall have for the conditions of equilibrium of a rigid body

$$P\cos\theta + P'\cos\theta' + \dots\dots = 0 \dots\dots(1),$$
$$P\sin\theta + P'\sin\theta' + \dots\dots = 0 \dots\dots(2),$$
$$Pa + P'a' + \dots\dots\dots\dots\dots\dots = 0 \dots\dots(3).$$

The equations (1) and (2) may be called the equations of equilibrium as regards *translation*, and are identical with those which hold for a single particle; equation (3) may be called the equation of equilibrium as regards *twisting* or *rotation*, and is peculiar to the case of a rigid body.

18. It is worthy of remark, that if a rigid body be capable of motion only about a certain fixed axis, the *three*

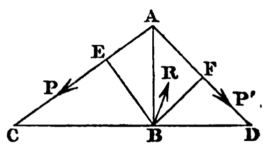
equations of the preceding article are reduced to *one;* for in this case any tendency to translation will be counteracted by a pressure on the axis, and the sole condition of equilibrium will be that the resultant moment of the forces about the axis shall be zero. Nevertheless we may in this case apply the three equations, if we desire, not only to determine the position of equilibrium for the body, but also to determine the pressure upon the axis: for let R be the pressure upon the axis, ϕ the angle which the direction of R makes with the line from which θ, θ'... are measured; then the equations (1), (2), of the preceding article, will become

$$R\cos\phi + P\cos\theta + P'\cos\theta' + \ldots = 0,$$
$$R\sin\phi + P\sin\theta + P'\sin\theta' + \ldots = 0;$$

and these will determine both R and ϕ.

19. We shall defer the full application of the equations of equilibrium until we have discussed, as we propose to do in the next chapter, what are called the Mechanical Powers, or the simplest cases of Machines; these might be considered merely as examples of the principles of this and the preceding chapter, but it will be convenient to group together (as is usual) in one chapter those problems which have a practical bearing, and then to collect in another such examples as may be considered chiefly theoretical and only useful as illustrations of Mechanical Principles. We shall however illustrate the meaning of the equations of this chapter by a few simple applications.

Ex. 1. AB is a vertical post moveable about a hinge at B; two men at C and D pull at the post by means of cords attached at A; given the height of the post and the lengths of the cords, compare the strengths of the men when AB remains vertical.



Let $AB = p$; $CA = l$; $DA = l'$; P, P' the forces exerted by the two men. Draw BE, BF perpendicular to AC, AD respectively; then for equilibrium we must have

$$\text{moment of } P \text{ about } B = \text{moment of } P' \text{ about } B,$$
$$\text{or } P \,.\, BE = P' \,.\, BF;$$

but by similar triangles ABC, BEC,

$$\frac{AB}{AC} = \frac{BE}{BC};$$

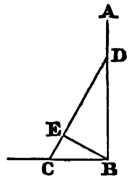
$$\therefore\ BE = \frac{AB \cdot BC}{AC} = \frac{p}{l}\sqrt{l^2 - p^2},$$

$$\text{similarly, } BF = \frac{p}{l'}\sqrt{l'^2 - p^2},$$

$$\therefore\ \frac{P}{P'} = \frac{l}{l'}\sqrt{\frac{l'^2 - p^2}{l^2 - p^2}}.$$

This formula gives us the ratio required.

Ex. 2. We see from the preceding investigation that the effect which a man can produce, by means of a rope attached in the manner described, is measured by the *moment* of the force which he exerts, not by the force itself. Let us illustrate this by inquiring under what circumstances a man can with the greatest advantage pull at a tree AB, by means of a rope of given length CD, attached to a point D in the tree.



Let F be the whole force which the man can exert; $CD = l$; $DCB = \theta$; draw BE perpendicular to CD; then the *moment of F about B*

$$= F \times BE = F \times BC \sin\theta,$$

$$= F \times l\cos\theta \sin\theta = \frac{Fl}{2}\sin 2\theta.$$

Now $\sin 2\theta$ has its greatest value when $2\theta = 90^\circ$, or $\theta = 45^\circ$. In this case $BC = BD$; or the man will pull to the greatest advantage, when the height of the point of attachment of the rope is equal to the man's distance from the tree: and the moment produced will be equal to that which would support the greatest weight the man can lift, suspended from the extremity of a rigid rod half as long as the rope.

Ex. 3. Let us inquire in Example 1, what will be the pressure sustained at the point B.

Let R be the pressure, and ϕ the angle which its direction makes with BD; also let $ACB = \theta$, $ADB = \theta'$; then we must have,

$$R\cos\phi - P\cos\theta + P'\cos\theta' = 0 \ldots\ldots\ldots(1),$$

$$R\sin\phi - P\sin\theta - P'\sin\theta' = 0 \ldots\ldots\ldots(2),$$

these equations correspond to (1) and (2) of Art. 17. The third equation of that article, or the *equation of moments*, we have already used in Ex. 1; we will however repeat it, making use of our present notation; it will be as follows,

$$Pp\cos\theta - P'p\cos\theta' = 0,\ (\text{since } BE = p\cos\theta,\ BF = p\cos\theta'),$$

$$\text{or } P\cos\theta - P'\cos\theta' = 0 \ldots\ldots\ldots\ldots(3),$$

equation (3) reduces (1) to the following,

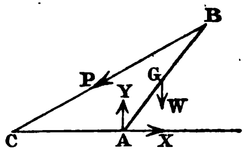
$$R\cos\phi = 0;$$

$$\therefore\ \phi = 90^0,$$

and then $R = P \sin\theta + P' \sin\theta'$, from (2).

Hence therefore the pressure at B will be a *vertical* pressure, and equal to the sum of the vertical resolved parts of P and P'. This is a conclusion which might have been anticipated; but it is desirable to see how the result arises from the general equations of equilibrium.

Ex. 4. AB is a heavy beam, moveable in a vertical plane about A, and inclined to the horizon at an angle of 45^0; required the force which must be exerted by a man standing at C, where $AC = AB$, to prevent the beam from falling.



Let the weight of the beam be W; this we may regard as a single vertical force acting at the centre of gravity G of the beam, and G will be the middle point of AB if we regard the beam as uniform. Let P be the force required; and $AB = a$.

Then for equilibrium the moments of P and W about A must be equal;

$$\therefore\ P \times a \sin ABC = W \times \frac{a}{2} \cos 45^0,$$

$$\text{or } P \sin 22^0\,30' = \frac{W}{2} \sin 45^0 = W \sin 22^0\,30' \cos 22^0\,30';$$

$$\therefore\ P = W \cos 22^0\,30',$$

$$= W\sqrt{\frac{1 + \cos 45^0}{2}} = W\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}},$$

which is the force required.

Ex. 5. In the preceding example, as in Ex. 3, the two equations of equilibrium which we have not used will give us the pressure sustained by the ground at the point A. Instead of calling this pressure R as in the former instance, and denoting by ϕ the angle which its line of action makes with the horizon, we will take X and Y to represent its horizontal and its vertical resolved part respectively. We shall then have, if we resolve horizontally and vertically,

$$X - P \cos BCA = 0,$$

$$Y - W - P \sin BCA = 0;$$

$$\text{but } BCA = 22^0\,30', \text{ and } P = W \cos 22^0\,30';$$

$$\therefore\ X = P \cos 22^0\,30' = W \cos^2 22^0\,30' = \frac{W}{2}(1 + \cos 45^0)$$

$$= \frac{W}{2}\left(1 + \frac{1}{\sqrt{2}}\right),$$

and $Y = W + P \sin 22^\circ 30' = W\,(1 + \sin 22^\circ 30' \cos 22^\circ 30')$

$$= W\left(1 + \frac{\sin 45^\circ}{2}\right) = W\left(1 + \frac{1}{2\sqrt{2}}\right).$$

These two expressions for the resolved parts X and Y entirely determine the magnitude and direction of the pressure; for if R and ϕ have the meanings above assigned to them, we have

$$R^2 = X^2 + Y^2, \text{ and } \tan\phi = \frac{Y}{X}.$$

Ex. 6. The general principles of equilibrium require, that the forces resolved in *any* two directions at right angles to each other should vanish, and that the moment of the forces about *any point* should also vanish. We will illustrate this by resolving the forces in the preceding example in the direction of AB and perpendicular to it, and by taking the moments about B. In considering the problem thus we must regard the beam AB as under the action of the four forces P, W, X and Y; and we will slightly vary the problem by supposing AB to make with the horizon a given angle θ; then we shall have

$$W \sin\theta + P\cos\frac{\theta}{2} - X\cos\theta - Y\sin\theta = 0 \ldots\ldots(1),$$

$$W\cos\theta - P\sin\frac{\theta}{2} + X\sin\theta - Y\cos\theta = 0\ldots\ldots(2),$$

$$Xa\sin\theta + W\frac{a}{2}\cos\theta - Ya\cos\theta = 0\ldots\ldots(3).$$

Multiply (1) by $\cos\theta$, and (2) by $\sin\theta$, and by subtraction there results,

$$P\left(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right) - X = 0,$$

$$\text{or } X = P\cos\frac{\theta}{2}.$$

Again, multiply (1) by $\sin\theta$, and (2) by $\cos\theta$, and by addition we have

$$W + P\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) - Y = 0,$$

$$\text{or } Y = W + P\sin\frac{\theta}{2};$$

therefore from (3)

$$\frac{W\cos\theta}{2} = Y\cos\theta - X\sin\theta,$$

$$= W\cos\theta + P\left(\sin\frac{\theta}{2}\cos\theta - \sin\theta\cos\frac{\theta}{2}\right),$$

$$= W\cos\theta - P\sin\frac{\theta}{2};$$

$$\therefore\ P = W\frac{\cos\theta}{2\sin\frac{\theta}{2}},$$

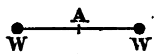
$$X = W\frac{\cos\theta}{2\tan\frac{\theta}{2}},$$

$$Y = W + \frac{W\cos\theta}{2}.$$

If $\theta = 45^0$, these results agree with those already obtained.

20. We will conclude this chapter by shewing how the principle of the lever may be established independently of the parallelogram of forces, and how it may then be made the basis of a system of Statics.

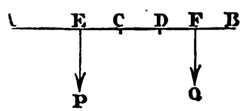
21. The demonstration depends upon the following axiom. *Two equal weights* W, W, *supposed to be connected by a rigid rod without weight will balance upon the middle point of the rod*



and will produce there a pressure equal to 2W. This axiom there is no difficulty in admitting, because, the weights being equal, there is no reason why one of them should descend rather than the other; and moreover, if the middle point of the rod be supported, the supporting point sustains the two weights, and therefore the pressure upon the point must be measured by the sum of the weights.

Hence it follows, that a uniform rod or cylinder will balance about its middle point, and will produce there a pressure equal to its weight; this is sometimes expressed by saying, that the statical effect of the rod or cylinder is the same as it would be if collected at its middle point. The truth of this immediately follows from the axiom just now enunciated, because we may consider the rod as cut up into any number of equal weights, and as each pair equidistant from the centre may be collected at the centre, the whole may be so collected.

22. Now let us take a uniform heavy rod AB, the weight of which is $P + Q$. This rod will balance about its middle point C.



Divide AB in D, so that

$$AD : DB :: P : Q,$$

then the weight of the portion AD is P, and that of DB is Q. Let E be the middle point of AD, and F of DB; then the statical effect of the rod AD is the same as that of a weight P suspended from E, and that of DB as that of a weight Q suspended from F. Hence the weights P and Q, suspended from E and F respectively, will balance about C; and we have now only to determine by geometry what is the relation of the two arms CE and CF.

We have $CE=AC-AE=BC-ED=DB-CE$,

$$\therefore DB=2CE;$$

$$\text{similarly, } AD=2CF;$$

$$\text{but } P : Q :: AD : DB, \text{ by construction,}$$

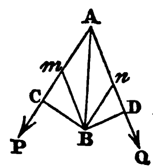
$$\therefore P : Q :: CF : CE;$$

$$\text{or } P.CE=Q.CF.$$

That is, the moments of P and Q about C must be equal; which is the principle of the lever.

The proposition thus proved for two weights is true for any two parallel forces, and we can easily deduce the case in which the forces are not parallel. This has in fact already been done in p. 25.

23. Assuming the principle of the lever, we can now prove the parallelogram of forces.



Let Am, An represent in magnitude and direction two forces P and Q acting at the point A: complete the parallelogram $AmBn$, and draw AB. Also draw BC, BD, perpendicular to Am, An produced. Now suppose AB to be a rigid rod or lever, moveable about B, and acted upon by the forces P and Q at A. Then

$$\frac{P}{Q}=\frac{Am}{An}=\frac{\sin mBA}{\sin mAB}=\frac{\sin nAB}{\sin mAB}=\frac{BD}{BC},$$

$$\text{or } P.BC=Q.BD;$$

therefore the forces P and Q would keep the lever at rest.

And since the resultant of P and Q would produce the same effect as P and Q together, it also acting at A would keep the lever at rest. But no single force acting at A can keep the lever at rest, unless it act in the direction

AB, in which case it will only produce a pressure upon *B* which we suppose to be fixed; hence *AB* is the *direction* of the resultant of *P* and *Q*.

Having thus proved the parallelogram of forces as regards *direction*, it may be extended to the *magnitude* precisely as in Art. 4, p. 51.

EXAMINATION UPON CHAPTER VI.

1. From the parallelogram of forces deduce the principle of the lever, the forces not being parallel.

2. Deduce the truth of the principle when the forces are parallel.

3. Shew that the principle of the lever if proved for a rigid rod may be extended to the case of any rigid body.

4. Define a *couple*, the *arm* of a couple, the *moment* of a couple.

5. The effect of a couple cannot be counteracted by the action of any single force.

6. The effect of a couple is not altered by turning its arm about one extremity through any angle in the plane of the forces.

7. The effect of two couples, the arms of which have a common extremity, and which tend to twist in the same direction, is the same, provided the moments be equal.

8. Shew how to find the resultant of any number of couples having the same plane.

9. Any system of forces, the directions of which lie in one plane, acting upon a rigid body, may be reduced to a single force and a single couple.

10. Investigate the conditions of equilibrium of a rigid body, the directions of the forces which act upon it lying all in one plane.

11. Prove, without assuming the parallelogram of forces, that two weights will balance upon a straight lever if their moments about the fulcrum be equal.

12. Assuming the principle of the lever, deduce the parallelogram of forces so far as the *direction* of the resultant is concerned.

13. If a man who can just lift 3 cwt., pull at a post as in Ex. 2, p. 71, by means of a rope twice as long as the post is high, find what horizontal force must be applied at its middle point to prevent it from falling.

14. In Ex. 3, p. 72, these words occur: "This is a conclusion which might have been anticipated." Explain this passage.

15. Solve the problem given in Ex. 4, p. 72, upon the supposition of the angle which AB makes with the horizon being 60^0, and supposing also that a weight equal to half the weight of the beam is suspended from B.

16. Under the circumstances supposed in the preceding example find the direction of the pressure at A, and construct the angle which determines it.

17. Three weights are suspended from the angular points of an equilateral triangle which is fixed in a vertical plane with one of its sides making an angle of 45^0 with the horizon; find the moment of the weights with respect to the centre of the triangle.

18. Two uniform beams of equal transverse section are fixed together by the extremities, so as to make with each other a right angle, and suspended from their point of junction; if one beam be twice as long as the other, find the position of equilibrium.

19. AB is a rod capable of turning freely about its extremity A, which is fixed; CD is another rod equal to $2AB$, and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force P acts at C in the direction CA: find the force which must be applied at D in order to produce equilibrium, the angle between the rods being given.

20. If a set of forces, acting at the angular points of a plane polygon, be represented in magnitude and direction by the sides taken in order, shew that their tendency to turn a body about an axis perpendicular to the plane of the polygon is the same through whatever point of the plane the axis passes.

CHAPTER VII.

ON MACHINES.

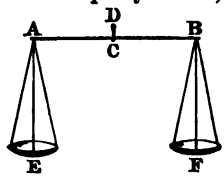
1. ANY contrivance by means of which force is transmitted from one point to another, or by means of which force is modified with respect to direction or intensity, is called a *machine*. We have already had a simple instance of a machine in the case of the lever. The oar of a boat, for example, is a machine; here the force applied at one end of the oar is converted into a force of propulsion at the rowlock; and in the same sense a poker, a crowbar, a pair of scissors, the human arm, may all be considered as machines. In this chapter we shall consider some other instances, and our purpose will be in each case to determine the conditions under which a certain force P, acting at one given point of a machine, will be in equilibrium with another force W, acting at another given point: P we shall usually call the *power*, and W the *weight*. Many machines are chiefly of practical use when they are in motion; thus in the case of the steam-engine, the expansive force of steam is applied to put machinery in motion; but all calculations connected with machines in motion belong to the science of Dynamics, not that of Statics, and we shall concern ourselves here only with examples of machines in equilibrium.

We shall begin by explaining two or three methods by which the property of the Lever is rendered available for the purpose of *weighing*.

2. *The Common Balance.*

Let AB be a rigid rod, CD a small rigid piece attached to its middle point and perpendicular to it, and let D be supported by a string or otherwise. E, F are two scales or pans of equal weight suspended by strings from A and

B. Then it is evident that if *A* and *B* be equally loaded, the *beam* *AB* will be horizontal; if not, the more heavily loaded scale will cause the extremity to which it is attached to preponderate. And thus by placing any given weight, as 1 lb. for instance, in the scale *E*, and putting such a quantity of any given substance into the scale *F* as shall allow of the beam *AB* being horizontal, we can weigh out a pound of that substance.



3. The preceding explanation represents the balance in its simplest form, and exhibits its principles: in practice many modifications and additional contrivances must be introduced; much skill has been expended upon the construction of balances, and great delicacy has been obtained. It would be beyond the scope of this book to describe all the features in the construction of firstrate balances, by means of which a degree of accuracy has been arrived at, which is truly wonderful: there are however two or three points to which it will be desirable to call attention.

The beam should be suspended by means of a knife-edge, that is, a projecting metallic edge transverse to its length, which rests upon a plate of agate or other hard substance. The chains which support the scales should be suspended from the extremities of the beam in the same manner.

The point of support of the beam should be at equal distances from the points of suspension of the scales; and when the balance is not loaded the beam should be horizontal.

To test the accuracy of a balance, first ascertain that the beam is horizontal when the balance is not loaded; then place two weights in the scales such that the beam shall be horizontal; lastly, change these weights into opposite scales, if the beam still remain horizontal the balance is a true one.

The chief requisite of a good balance is what is termed *sensibility;* that is to say, if two weights which are very nearly equal be placed in the scales, the beam should vary *sensibly* from its horizontal position. In order to produce this result two conditions should be satisfied; (1) the point

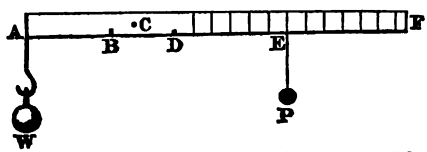
of support of the beam and the points of suspension of the scales should be in the same straight line; the consequence of this will be that two equal weights in the scales will produce a resultant through the point of support, they will therefore have no effect whatever in twisting the beam, and the deviation from horizontality will be the same for a given *difference* of weights however great the weights themselves may be; (2) the point of support should be very near the centre of gravity of the beam, and a little above it; the nearer these two points are to each other the greater will be the sensibility, for the weight of the beam acting at its centre of gravity must be in equilibrium with the small difference of the weights acting at one end of the beam, and this difference of the weights will act at a greater mechanical advantage the nearer the centre of gravity of the beam is to the fulcrum.

If the sensibility of a balance be very great, the addition of a small weight to either scale will cause the beam to oscillate, and some time will elapse before it attains its position of equilibrium; on this account the beam is sometimes furnished with a pointer and a graduated arc of a circle; if the pointer oscillates through equal arcs on opposite sides of the point which corresponds to horizontality, we may be satisfied that the scales are equally loaded, without waiting to ascertain whether the beam will ultimately rest in a horizontal position.

4. The common balance requires a series of weights in order to render it practically useful, but there is another kind of weighing machine in which one and the same weight is made use of in all cases. This is the instrument known as the Roman or Common Steelyard.

The Common Steelyard.

Let AF be a rigid bar moveable about a horizontal pivot at C; and from A let the weight W which we desire



to measure be suspended. P is a given moveable weight, which can be suspended from any point E of the bar between

C and F; and it is evident, from the principle of the lever, that the larger is W the further must the point of suspension E be from C, in order that the steelyard may be horizontal. Suppose then a certain weight suspended at A; the point of suspension of P must be shifted until the steelyard is horizontal, and the bar is so *graduated* that by looking at the number which is nearest to E we can at once ascertain the weight of W.

5. The process of graduating the steelyard deserves attention.

PROP. *To graduate the common Steelyard.*

Remove the weights P and W, and suppose that under these circumstances the arm CF of the steelyard preponderates; find, by trial, the point B, such that if P be suspended from B the steelyard will be horizontal; take $CD = CB$, then the moment of the weight of the steelyard about C is the same as that of P suspended from D. Now let W hang from A, and P from any point E, then for equilibrium we must have

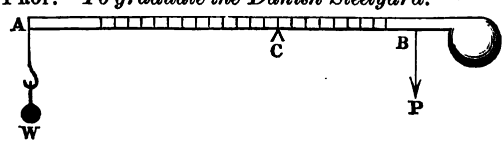
$$W \times AC = P \times CD + P \times CE = P \times BE;$$

$$\therefore BE = \frac{W}{P} \cdot AC.$$

Suppose that $P = 1$ lb.; and make W successively $= 1$ lb., 2 lbs., 3 lbs., &c., then the values of BE will be AC, $2AC$, $3AC\ldots$, and these distances must be set off, measuring from B, and the points so determined marked 1 *lb.*, 2 *lbs.*, 3 *lbs.*, *&c.*

6. Another form of this balance is that which is called the *Danish Steelyard*, in which the weight is fixed to the beam and the fulcrum is moveable. This is, for the greater number of purposes, not so convenient a construction as the preceding; it is however not inconvenient for weighing small weights, when no great accuracy is required; letter-balances are sometimes made upon this principle.

PROP. *To graduate the Danish Steelyard.*



Let B be the point on which the instrument would

balance, if no weight were suspended at A; and when the weight W is suspended at A let C be the place of the fulcrum; also let P be the entire weight of the instrument, which may be supposed to be collected at B, or which, in other words, will produce a downward pressure at B equal to P. Then for equilibrium we must have

$$W \times AC = P \times BC = P(AB - AC);$$

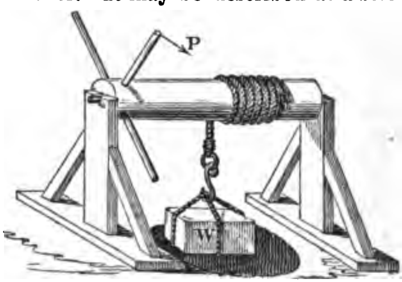
$$\therefore AC = \frac{P}{W+P} \cdot AB.$$

Hence, making $W = 1$ lb., 2 lbs., 3 lbs. ... successively, we shall be able to mark upon the steelyard the corresponding positions of the fulcrum; and when the beam is thus graduated we shall be able to ascertain the weight of any given body suspended from A, by observing the mark of graduation which is nearest to the fulcrum.

7. It will be seen that the distances between the successive marks of graduation on the common steelyard are equal, but on the Danish unequal. In fact, the distances of the successive marks of graduation from A, the extremity of the beam which supports W, in the common steelyard form an *arithmetical* progression, in the Danish they form an *harmonical*.

8. The principle of the lever may be conveniently applied for the purpose of lifting or sustaining great weights; this is done by means of a *windlass* or *capstan*.

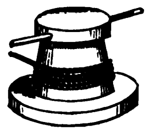
The *windlass* is used for such purposes as that of raising an anchor. It may be described as a strong cylin-



drical beam, moveable about a horizontal axis, the extremities being inserted into two strong upright pieces in which they are capable of turning freely. One end of a rope is

coiled partially round the windlass, and to the other end is attached the anchor or the weight to be raised; a number of apertures are made in the windlass perpendicular to its axis, and in these are inserted short bars called *handspikes;* by means of these it is evident that the windlass may be made to revolve, and when by its revolution a handspike is brought inconveniently low it is taken out and reinserted in a more convenient place. The windlass in the figure is represented with fixed bars, instead of handspikes, which in some applications of the machine is a more convenient arrangement.

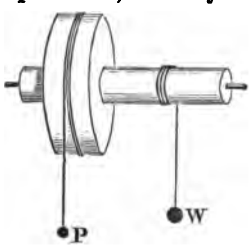
9. Some inconvenience arises from the necessity of changing the position of the handspikes; this is avoided in the *capstan*, the principle of which is the same as that of the windlass, but the axis is vertical, and a person may therefore by moving his own position cause the capstan to revolve without changing the point of insertion of the handspike.



10. In both the preceding cases the mechanical advantage gained depends of course upon the length of the handspike, which however is limited by considerations of practical convenience. The actual relation between the *power* and *weight* upon machines of this kind will be seen in the investigation of these conditions for the machine known as

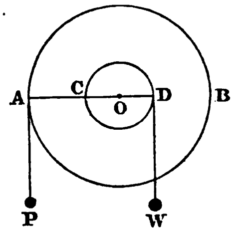
The Wheel and Axle.

This machine consists, in its simplest form, of two cylinders having their axes coincident; the two cylinders forming one rigid piece; the larger is called the *wheel*, the smaller the *axle*. The cord by which the weight is suspended is fastened to the axle and coiled round it; the power may be supposed to act in like manner by means of a cord coiled round the wheel, as in the figure; or the power may act by means of a handle, as in the case of the common well and bucket.



11. *To find the ratio of* P *to* W, *when there is equilibrium upon the Wheel and Axle.*

Let AB, CD represent sections of the wheel and axle respectively, and O their common centre; P and W the power and weight, acting by means of strings at the circumference of the wheel and axle respectively.



For simplicity's sake P, W, and the arms at which they act, are in the figure represented in the same plane.

From the common centre O draw OA, OD to the points at which the cords supporting P and W touch the circumferences of the wheel and axle respectively; these lines will be perpendicular to the directions in which P and W act; hence, by the principle of the lever, or in other words taking moments about O,

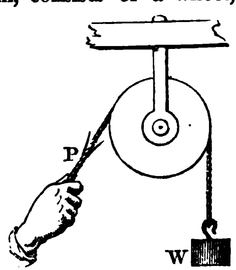
$$P \times AO = W \times OD,$$

$$\text{or } \frac{P}{W} = \frac{OD}{AO} = \frac{\text{radius of axle}}{\text{radius of wheel}}.$$

It is evident that the larger the radius of the wheel, the greater will be the mechanical advantage, that is, the smaller will be the power P necessary to support or to raise any given weight W.

12. *The Pully.*

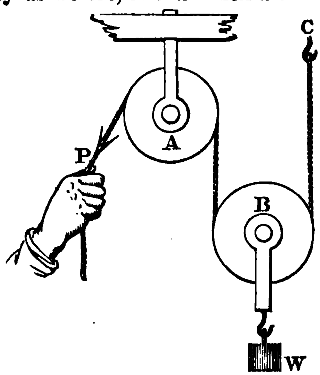
The Pully, in its simplest form, consists of a wheel, capable of turning about its axis, which may be either fixed or moveable. A cord passes over a portion of the circumference; if the axis of the pully be fixed, the only effect of the pully is to change the direction of the force exerted by the cord, and in this case no mechanical advantage is gained so far as the intensity of the force is concerned. Nevertheless the contrivance may be very convenient; for example, if we wish to raise a heavy weight, we can frequently do so most conveniently by attaching to it a cord which passes



over a fixed pully, as in the figure; the effort, which must be exerted in this case to raise the weight, is the same as that which would be exerted to raise it without the intervention of the pully.

But suppose we modify the preceding contrivance as follows.

Let A be a fixed pully as before, round which a cord passes, and let this cord, instead of being made fast to the weight W, pass round a moveable pully B from which the weight depends, and then be made fast to a fixed point C. In this case, not only is the direction of the force changed, so that a person pulling *downwards* raises the weight, but also the force which he will have to exert will be equivalent

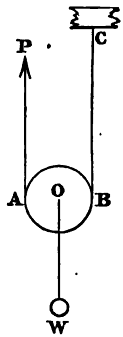


to only half the weight raised; for instance, a weight of 1 lb. suspended at the *power end* of the cord will raise a weight of 2 lbs.; and we shall find that by various combinations of pullies still greater advantage can be gained; in fact, by a sufficiently complicated system of pullies we can make a given force support any weight however large. We shall investigate the relation of P to W in the case of the single moveable pully, and also in the case of several complicated systems; these systems may be multiplied to any extent, but the method of finding the relation of P to W will apply *mutatis mutandis* to all.

In practice the pullies are made of wood or metal, and are therefore heavy bodies, whose weight ought in strictness to be taken into account; but for simplicity's sake we shall neglect the weight of the pullies, as for like reasons we shall that of the cord which passes round them. We shall also suppose the portions of cord to be parallel and vertical.

13. *To find the ratio of the Power to the Weight in the single moveable Pully.*

Let O be the centre of the pully, which is supported by a cord passing under it and attached to a fixed point C at one end, and stretched by the force P at the other. Suppose the weight to be suspended from the centre O.



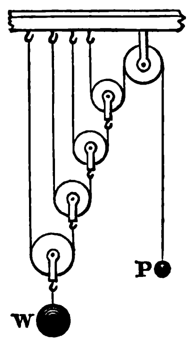
Then the pully with its depending weight W is supported by two strings AP and BC; the tension of the former is P, because by hypothesis the force P acts at the end of it; that is to say, the string AP exercises a supporting force upon the pully equal to P. Now the string BC which acts upon the other side of the pully is similarly circumstanced to AP, and must therefore exert an equal supporting force upon the pully. Hence on the whole the pully is acted upon by two equal forces, each equal to P, upwards, and by the weight W downwards, and therefore we must have

$$2P = W,$$

$$\text{or } \frac{P}{W} = \frac{1}{2}.$$

14. *To find the ratio of the Power to the Weight, in a system of Pullies, in which each pully hangs by a separate string.*

This system is represented in the figure, and is usually spoken of as the *First System of Pullies.*



By the property of the single pully the tension of the string which supports the lowest pully will be $\frac{W}{2}$. The tension of the string which supports the lowest but one will be $\frac{W}{2^2}$; and so on. Let there be n pullies, n being any number; then the tension of the string which supports the n^{th} pully will be $\frac{W}{2^n}$; but this must be equal to P, since the tension of the string which supports the n^{th} pully is produced by the force P;

$$\therefore\ P=\frac{W}{2^n},\ \text{or}\ \frac{P}{W}=\frac{1}{2^n}.$$

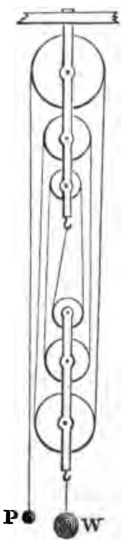
It will be seen, that in this system the mechanical advantage gained increases very rapidly with the number of pullies; thus if

$n=2$, a weight of 1 lb. will support 4 lbs.
$n=3$, ... 8 lbs.
$n=4$, ... 16 lbs.

and so on.

15. *To find the ratio of the Power to the Weight, in a system of Pullies, in which the same string passes round all the Pullies.*

This system will be understood from the figure, and is known as the *Second System of Pullies.*



There are two blocks, the lower one moveable, the upper one fixed, and each containing a number of pullies. The same string goes round all the pullies, and therefore the tension throughout will be the same, and equal to the power P. Let n be the number of strings at the lower block, then the sum of their tensions will be nP, and we shall have

$$nP=W,\ \text{or}\ \frac{P}{W}=\frac{1}{n}.$$

The mechanical advantage does not, in this system, increase so rapidly with the increase of the number of pullies as in the previous system; but on many accounts it is practically more convenient.

16. *To find the ratio of the Power to the Weight, in a system of Pullies, in which all the strings are attached to the weight.*

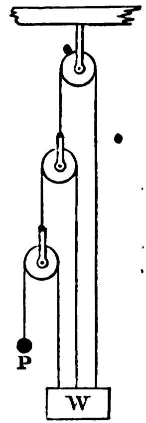
This system is represented in the figure, and is known as the *Third System of Pullies.*

The tension of the string which supports P is P; that of the next string is $2P$, by the property of the single pully; that of the next is 2^2P; and so on. Let there be n strings, then the tension of the last is $2^{n-1}P$; and the sum of all the tensions is

$$(1+2+2^2+\ldots+2^{n-1})P,\ \text{or}\ (2^n-1)P.$$

But the sum of all the tensions must be equal to W, since the strings support W;

$$\therefore\ (2^n - 1)P = W, \text{ or } \frac{P}{W} = \frac{1}{2^n - 1}.$$

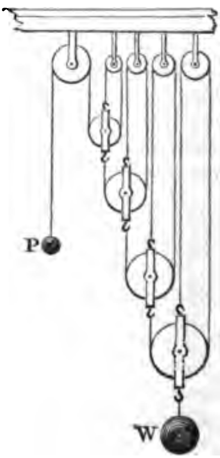


For instance,

if $n = 2$, a weight of 1 lb. will support 3 lbs.
$n = 3$, ... 7 lbs.
$n = 4$, ... 15 lbs.

It will be seen, that the gain of mechanical advantage in this system is nearly the same as in the first system of pullies.

17. The principles upon which the relation of P to W has been determined in the preceding articles are (as has been already remarked) applicable to all systems of pullies, however complicated. A rule may be given, as follows, but its meaning will be best seen by applying it to examples. Begin at the *Power-end* of the system, then the tension of the string which supports P will be equal to P throughout; against each of the parallel portions of this string write P; now proceed to the next string, find what its tension is by observing how many strings, each having the tension P, produce it; write the expression for its tension against each parallel portion of it; and so with the next string. When the tension of each string of the system has been written down, it is easy to see how many of them support W, and by adding their tensions together we have the relation between P and W required.



18. We will illustrate this by a rather complicated system, represented in the figure. The P string occurs three times, and produces a tension $3P$ in the next string; this again occurs three times, and therefore produces a tension 3^2P or $9P$ in the next; and so on. If we have three pullies, as in the figure, the

result will be

$$27P = W; \text{ or } \frac{P}{W} = \frac{1}{27}.$$

If more generally we take n pullies, we have

$$3^n P = W, \text{ or } \frac{P}{W} = \frac{1}{3^n}.$$

19. *The Inclined Plane.*

By an inclined plane is meant a plane inclined to the plane of the horizon, and the angle which it makes with the plane of the horizon is called the *inclination* of the plane.

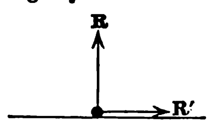
If a weight be placed upon a *horizontal* plane it will rest in the position in which we place it, because the effect of a body's weight is in this case only to make it press against the plane, which returns the pressure; but if we place a weight upon a smooth *inclined* plane, unless it be supported, it will slide down, for, in this case, the tendency which the body has to descend is not entirely checked by the plane. In practice, a body will remain at rest upon the surface of a plane of considerable inclination, but this arises from the fact that in practice all bodies are more or less *rough*, and the roughness of the inclined plane will be sufficient to prevent a weight from sliding down it, if the inclination be not very great; we shall say something more upon this subject when we come to the general consideration of *friction;* at present we shall suppose that the inclined plane is perfectly smooth, that is, that it is incapable of offering any resistance to the sliding of a body along its surface.

The problem in the case of the inclined plane is this, to determine what force P, acting in a given direction, will support a given weight W, resting upon a plane of given inclination. It may perhaps be asked, how this problem properly comes under the head of *machines;* but it will be seen by reference to our definition of a machine in Art. 1 (p. 78), that the inclined plane is rightly so regarded, for it supplies us with the means of modifying the effects of a given force. Moreover, an example will shew that the inclined plane may be used as a means of assisting human strength, in the same manner as the lever or the pully: for let it be required to raise a cask of wine from a cellar, then we may

either roll the cask to the side of the cellar, and extract it by means of a crane and pully, or we may lay down some planks at a moderate inclination and drag up the cask upon them.

20. Before we proceed to find the relation of P to W upon the inclined plane, we must make an important remark respecting the pressure exerted by a plane upon a body which rests upon it. If a particle rests upon a horizontal plane the forces which keep it at rest are *two;* viz. the weight of the particle *downwards*, and a certain pressure caused by the plane *upwards*, and these must be equal, otherwise the particle could not be at rest; hence in considering the equilibrium of such a particle we may dismiss all thought of the plane, and say that the particle is kept at rest by its own weight W acting vertically downwards, and a pressure W acting vertically upwards. Now let us consider what will be the mechanical effect of a smooth plane, which is not horizontal, upon a particle made to rest upon it. Its effect will be to produce a pressure upon the particle; and there will be two questions, what will be the direction of this pressure, and what will be its magnitude?

(1) For the direction, we can at once conclude that the pressure must be perpendicular to the plane; because the plane is by hypothesis *smooth*, and by the term *smooth* we mean that it is incapable of offering any resistance to the motion of a particle along its surface. To make this more clear, suppose the pressure exerted by the plane to be in any direction whatever: then since a force may always be *resolved* into two at right angles to each other, let this pressure be resolved into two, one perpendicular to the plane, which call R, and one parallel to the plane, which call R': now the force R' will manifestly tend to make the particle move along the surface of the plane; but this is contrary to the definition of a *smooth* plane, therefore $R'=0$; and hence the only force exerted by the plane is a force R in the direction perpendicular to it. But



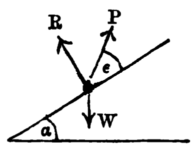
(2) What will be the magnitude of R? This we have not sufficient data to determine; it will vary in different cases, and we cannot determine it until we have all the circumstances given.

Hence, in any given problem, we may consider the effect of a smooth plane to be this, *to produce a force or pressure upon a particle in contact with it, in the direction perpendicular to it, but of unknown magnitude, and which we must therefore denote by a symbol for an unknown quantity, such as R.*

This being premised, we proceed

21. *To find the ratio of the Power to the Weight, when there is equilibrium on the Inclined Plane.*

Let α be the inclination of the inclined plane to the horizon; R the pressure of the plane on the weight W, which pressure will be perpendicular to the plane; and, to take the most general case, let the direction of the power P make an angle ϵ with the plane.



Then resolving the forces parallel and perpendicular to the plane, we have

$$P \cos \epsilon - W \sin \alpha = 0 \text{ .... } (1),$$
$$R + P \sin \epsilon - W \cos \alpha = 0 \text{ .... } (2).$$

$$\text{Hence, } \frac{P}{W} = \frac{\sin \alpha}{\cos \epsilon}, \text{ from (1).}$$

Equation (2) gives us the pressure upon the plane; thus

$$\begin{aligned} R &= W \cos \alpha - P \sin \epsilon, \\ &= W \cos \alpha - W \frac{\sin \alpha}{\cos \epsilon} \cdot \sin \epsilon, \\ &= \frac{W}{\cos \epsilon} (\cos \alpha \cos \epsilon - \sin \alpha \sin \epsilon), \\ &= W \frac{\cos (\alpha + \epsilon)}{\cos \epsilon}. \end{aligned}$$

There are two particular cases, which are worthy of notice.

(1) Suppose the power acts parallel to the plane, then the equations become

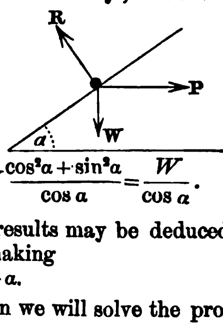
$$P - W \sin \alpha = 0,$$
$$R - W \cos \alpha = 0;$$
$$\therefore \frac{P}{W} = \sin \alpha, \text{ and } \frac{R}{W} = \cos \alpha.$$

If we regard the inclined plane as the hypothenuse of a right-angled triangle, having its two sides respectively

horizontal and vertical, and if we take the hypothenuse to represent the magnitude of W, then these results shew that the vertical side represents P, and the horizontal side represents R. This truth is exhibited to the eye, with much simplicity and beauty, by the apparatus of Professor Willis, referred to in the note on page 11.

It will easily appear that if we have a double inclined plane, that is, such a plane as would be represented by an obtuse-angled triangle having its base horizontal, and if we place upon the two sides two weights connected by a string passing over the vertex, then in order that there may be equilibrium the weights must be proportional to the lengths of the sides upon which they respectively rest. This property of the inclined plane was demonstrated by Stevinus of Bruges, who died in 1633, and was made by him the foundation of the theory of forces. The demonstration will be given at the end of this chapter.

(2) Suppose the power acts horizontally; then the equations will be,



$$P\cos a - W\sin a = 0,$$
$$R - P\sin a - W\cos a = 0;$$
$$\therefore \frac{P}{W} = \tan a,$$

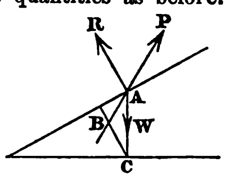
$$\text{and } R = W\cos a + W\frac{\sin^2 a}{\cos a} = W\frac{\cos^2 a + \sin^2 a}{\cos a} = \frac{W}{\cos a}.$$

It may be remarked that these results may be deduced from those of the general case by making

$$\epsilon = 0 \text{ and } \epsilon = -a.$$

22. For the sake of illustration we will solve the problem of the inclined plane in another way.

Let a, ϵ, R represent the same quantities as before. Let A be the point of the plane at which the weight rests; draw AC vertical, and from C draw CB in a direction perpendicular to the inclined plane, to meet the line of P's action in B. Then the sides of the triangle ABC, being parallel to the directions of the forces P, R, W, may be taken to represent these forces (Art. 6, p. 52). Hence



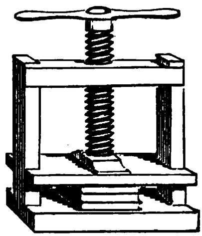
$$\frac{P}{\sin C} = \frac{W}{\sin B} = \frac{R}{\sin A}.$$

But $A = 90^0 - \alpha - \epsilon$, $B = 90^0 + \epsilon$, $C = \alpha$;

$$\therefore \frac{P}{\sin \alpha} = \frac{W}{\cos \epsilon} = \frac{R}{\cos (\alpha + \epsilon)}.$$

These are the same results as those already obtained.

23. The last machine which we shall consider is *the Screw.* This machine in combination with the lever is of great practical utility, as, for instance, in the case of a bookbinder's press, in which a considerable pressure is required, and may be by this means produced with great facility; and there are numberless other examples.



The Screw may be described as an inclined plane wrapped round a cylinder, or as a cylinder having on its surface a projecting thread in all parts at the same given angle to the horizon: take any solid body of a cylindrical form, as, for instance, a ruler, a pencil; take a piece of paper ABC in the form of a right-angled triangle, having the right angle at C; place BC upon the cylinder, so as to be parallel to its axis, and wrap the paper closely upon the cylinder, then the hypothenuse AB will mark out the thread of a screw. The form of the thread is different in different cases; it may be such as in fig. I., or such as in fig. II.; but this is a matter into which we shall not enter, and we shall consider the thread only as the surface of an inclined plane wrapped round a cylinder as before described.

Fig. I.

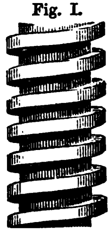


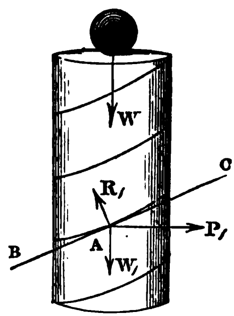
Fig. II.

The screw is applied as follows: the cylinder bearing the thread fits into a block pierced with an equal cylindrical aperture, upon the inner surface of which is cut a groove, the exact counterpart of the thread of the screw; hence the screw can only be made to move in the block by revolving about its axis. Suppose the axis of the

screw to be vertical, and a weight W to be placed upon it, then the screw would descend, unless prevented from doing so by another force; this force we will suppose to be supplied by the power P acting in a horizontal direction, at the extremity of an arm of given length. In practice there must necessarily be considerable friction between the thread and the groove, but this we shall not consider, because it would complicate the problem.

24. PROP. *To find the ratio of the Power to the Weight in the Screw.*

Let the power P act at an arm a, and let r be the radius of the cylinder, α the inclination of the thread to the horizon.



Consider the equilibrium of any point A of the thread; suppose the portion of the thread on each side of A to be unwrapped, so as to assume the position of a straight line BC, inclined at an angle α to the horizon; then we may consider the point A as supported upon a plane of inclination α, and acted upon by the pressure of the plane, which call R_1, a certain horizontal force caused by P, which call P_1, and a portion of the weight W, which call W_1; hence we shall have by resolving the forces in the direction BC,

$$W_1 \sin \alpha = P_1 \cos \alpha.$$

In like manner, if we considered the equilibrium of any other point of the thread, we should have

$$W_2 \sin \alpha = P_2 \cos \alpha;$$

and so on.

Hence, taking account of all the points of the thread, and adding all the equations together, we shall have

$$(W_1 + W_2 + W_3 + \ldots) \sin \alpha = (P_1 + P_2 + P_3 + \ldots) \cos \alpha.$$

But $W_1 + W_2 + W_3 + \ldots =$ the whole weight supported $= W$. Also, $P_1 + P_2 + P_3 + \ldots =$ the whole horizontal force supposed to act at the circumference of the cylinder, that is, at an arm r. But the horizontal pressure is caused by P acting at an arm a; hence, by the principle of the lever,

$$(P_1+P_2+P_3\ldots)\,r=Pa;$$

$$\therefore\ W\sin a=\frac{Pa}{r}\cos a,$$

$$\text{or } \frac{P}{W}=\frac{r}{a}\tan a.$$

We may put this result in a more convenient form thus;

$$\frac{P}{W}=\frac{2r\tan a}{2\pi a},$$

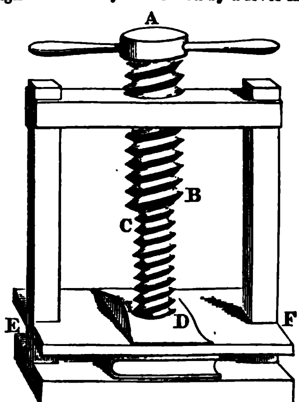
$$=\frac{\text{the vertical distance between two threads}}{\text{circumference of circle described by } P}.^{*}$$

25. This investigation may be presented under a rather different form as follows:

The weight W may be conceived as a body acted upon by these three forces, its own weight W vertically downwards, a certain force in a direction perpendicular to the thread of the screw, which will be the resultant of the

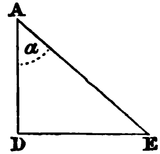
* It appears from this result that the power of the screw depends *cæteris paribus* upon the fineness of its thread; but if the thread be made too fine there is danger of fracture. This difficulty is overcome in a very ingenious manner by a machine known as *Hunter's Screw.*

AB is a screw which passing through a block may be worked by a lever in the usual manner; but instead of pressing immediately upon the board *EF*, the screw *AB* acts upon another screw *CD* which enters the former by means of an interior thread cut to fit it. The screw *AB* is slightly coarser than the screw *CD*, so that when *AB* descends through a given space, *CD* ascends through a space not quite so great, and the board *EF* is pressed through a space equal to the difference of the spaces respectively passed through by the two screws. It is not difficult to see, that as in the common screw the mechanical advantage depends upon the distance between the threads, so in this compound screw the mechanical advantage will depend upon the difference of these distances for the two component screws.



Hence by means of two coarse screws adapted to each other as above explained, we can produce the effect of one fine screw. The same principle may be applied by cutting two screws upon the same cylinder, and passing these through two nuts or blocks, which are capable of approaching each other, but are not allowed to revolve.

pressures at the different points of the thread, and lastly, a horizontal force which we will call Q.



Now draw AD vertical, AE in a direction perpendicular to the thread of the screw, and DE horizontal; then the sides of the triangle ADE, being parallel to the three forces just now described, may be taken to represent them;

$$\therefore \frac{Q}{W} = \frac{DE}{AD} = \tan a.$$

But Q is a force which at an arm r is in equilibrium with P at an arm a,

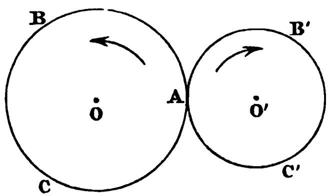
$$\therefore Q \times r = P \times a, \text{ or } Q = P\frac{a}{r},$$

$$\therefore \frac{P}{W} = \frac{r}{a} \tan a,$$

which is the result previously obtained.

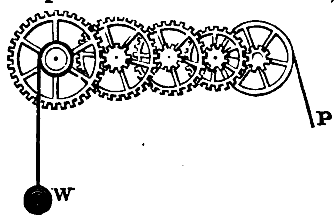
26. In this chapter on Machines I have omitted the consideration of *toothed wheels;* I have done so because they involve no distinct mechanical principle, and, as being chiefly useful for the transmission and modification of motion in machinery, belong rather to the subject of *Mechanism* than to that of *Mechanics*.

In order to understand the construction of toothed wheels let BAC, $B'AC'$ be two wheels, lying in the same plane, turning about centres O, O', and being in contact at A. And suppose that the friction between the surfaces of these two wheels is so great that they cannot slide one upon the other; then if we turn the wheel BAC in the direction of the arrow marked upon it, it is evident that the wheel $B'AC'$ must also turn, but in the opposite direction, that is, in the direction indicated in the figure by the arrow upon $B'AC'$. Thus the motion of BAC produces motion in $B'AC'$; and $B'AC'$ may in like manner be made to *drive* another wheel, and so on. Moreover, it is easy to see that if the magnitude of BAC be given, and the wheel be made



to turn at a given rate, the wheel $B'AC'$ will turn more slowly or more rapidly than BAC in exact proportion as its circumference (or its diameter) is greater or less than that of BAC. For instance, suppose the diameter of BAC to be two feet, and that of $B'AC'$ to be one foot; then if BAC be made to turn 30 times in a minute, $B'AC'$ will turn 60 times, and so on. Hence wheels connected as here described may be made both to transmit and also to modify motion.

But practically it is not possible to construct wheels, the surfaces of which shall drive accurately by means of friction only; hence the device of *teeth*, that is of alternate projections and hollows upon the surface of the wheels. The figure will shew

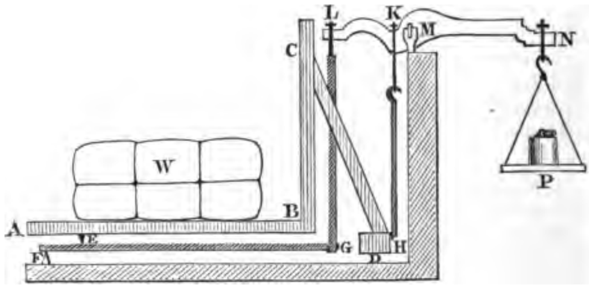


at once, after what has been said, the action of the teeth. Mechanically speaking a train of wheel-work is only a succession of levers; and if the number and magnitude of the wheels be given, there is no difficulty whatever in determining the power P, which, acting at the circumference of one extreme wheel of the train, will be in equilibrium with the weight W acting upon the other extreme wheel.

27. In addition to the simple machines considered in this chapter there are many others, contrived for a variety of purposes, as weighing, raising weights, &c. We will illustrate the statical principles already explained by applying them to a weighing machine of somewhat complicated character.

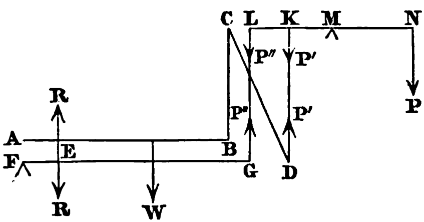
A platform AB, upon which the body which we wish to weigh is placed, is supported at one end by the piece BC which is connected (as shewn in the figure) with D, so that $ABCD$ is one rigid piece. AB rests at E upon a lever FG, (concerning which more presently,) and at the other extremity it is supported by means of a rod HK connected with the piece D. The lever FG, which has F for its fulcrum, is also supported by a rod GL; and both HK, and GL, are finally supported by a lever $LKMN$, which having M for

its fulcrum, carries at its other extremity a scale in which is placed the weight P which is to be in equilibrium with W.



This is a general description of the machine in question; now let us reduce it to its simplest statical form.

The figure below represents the statical problem; the letters in it correspond to those in the preceding description; and it will be seen that the machine consists essentially of three portions; the platform $ABCD$, the lever FG, and the lever $LKMN$; the two rods GL, DK only serve to connect these parts. Let us consider the forces to which they are severally subject.



(1) The platform $ABCD$ is acted upon by the downward pressure of the weight W, and by two upward pressures at the two points of support, which (as we do not know their values) we will denote by R and P' respectively.

(2) The lever FG is acted upon by a downward force at E, which must be equal and opposite to R, and by an upward force at G, which we will denote by P''. There is, of course, besides these two forces, the pressure upon the fulcrum, which it will not be necessary for us to consider.

(3) The lever $LKMN$ is acted upon at one extremity by the force P, and on the other side of the fulcrum there are two pressures at the point of support of the rods DK, GL, which must be equal to P' and P'' respectively.

We can now write down the equations of equilibrium for the machine.

For the platform $ABCD$ we must have, for the equilibrium of the vertical forces,

$$W = R + P' \quad\dots\dots\dots\dots\dots\dots (\alpha).$$

For the lever FG, taking moments about the fulcrum F,

$$R \, . \, FE = P'' \, . \, FG \quad\dots\dots\dots\dots\dots\dots (\beta).$$

For the lever $LKMN$, taking moments about the fulcrum M,

$$P.MN = P'.MK + P''.ML \dots\dots (\gamma).$$

If between (α) and (β) we eliminate the unknown quantity R, we have

$$W.FE = P'.FE + P''.FG \dots\dots (\delta).$$

We have now reduced the problem to the two equations (γ) and (δ), which involve two unknown forces P' and P''; and a little consideration will convince us that we have not omitted any equation essential to the solution of the problem. Hence in its present form the problem is indeterminate; but there is a peculiarity in the construction, which has been omitted hitherto, and in virtue of which the indeterminateness of the problem disappears. The machine is so arranged that the following proportion holds amongst the lengths of the arms of the levers,

$$FE : FG :: MK : ML.$$

If we introduce this condition into equations (γ) and (δ), we have

$$\frac{P\,.\,MN}{MK} = P' + P'' \frac{ML}{MK} = P' + P'' \frac{FG}{FE} = W;$$

$$\therefore \frac{P}{W} = \frac{MK}{MN}.$$

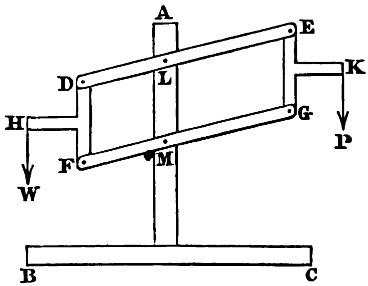
This is the required relation between P and W. We now see the advantage of the machine: suppose, for instance, that $MN = 10 \times MK$, then $\frac{P}{W} = \frac{1}{10}$, or a weight of 10 lbs. put into the P scale will serve to weigh 100 lbs. placed upon the platform. Machines upon this and similar

constructions are frequent at railway stations, and are very convenient for weighing heavy goods.

It will be seen from the investigation, that it is indifferent at what point or points of the platform the pressure W takes place.

28. Another machine worthy of notice is the balance so extensively used in retail trades, and which is known as *Roberval's balance.* The construction of this balance, viewing it merely in its mechanical principles, may be described thus.

ABC is a firm vertical stand resting upon a fixed horizontal base; DE, FG are two equal bars working about pivots similarly situated with respect to their lengths, at L and M; DFH, EGK are two T-shaped pieces connected with the bars DE, FG, as in the figure, by pivots; the consequence of which is, that if the system be made to assume different positions, DF, EG will always be vertical. Now suppose two weights P and W suspended from K and H respectively; then in order to investigate the relation of P to W we might consider, as in the case of the weighing machine last described, the equilibrium of the separate members of the balance successively; but, without doing so, the result may perhaps be rendered intelligible by general reasoning. Consider the weight W; it is supported by the piece DFH, which again is supported at D and F; now the effect of this must be to produce two vertical pressures at D and F which shall together be equal to W; it does not signify to our purpose how much one point bears and how much the other, nor what horizontal pressure there may be at either, (since whatever horizontal pressure there may be at one of them, there must be an equal and opposite horizontal pressure at the other,) the pressure must on the whole be W, or we may say that there is a pressure W in the direction of the rod DF; so that, as far as the equilibrium of the bar DE is concerned, we may regard W as



suspended from D; and in like manner we may regard P as suspended from E. Consequently the condition of equilibrium will be

$$\frac{P}{W}=\frac{DL}{EL}.$$

Or if the arms DL, EL be equal, then we must have $P=W$. It is to be noticed that the ratio of P to W is quite independent of the distance of either from the central axis of support; a circumstance which gives this balance great practical advantages.

29. We will conclude this chapter by giving the proof of the property of the inclined plane by Stevinus, to which reference was made in p. 92. The proof shall be given precisely as he gives it himself in his treatise on Statics. In the first book of this treatise he considers the theory of *direct* and *oblique* weights; the fundamental proposition of the theory of the former is the doctrine of the lever, demonstrated after the manner of Archimedes*; having solved a variety of problems involving the principle of the lever, he says, "Hitherto have been declared the properties of direct weights; here follow the properties and qualities of oblique, the general foundation of which is contained in the following theorem."

THEOREM. *If a triangle have its plane perpendicular, and its base parallel, to the horizon; and upon the two sides be placed two equal spherical weights; as the right side of the triangle is to the left, so will be the power of the left hand weight to that of the right.*

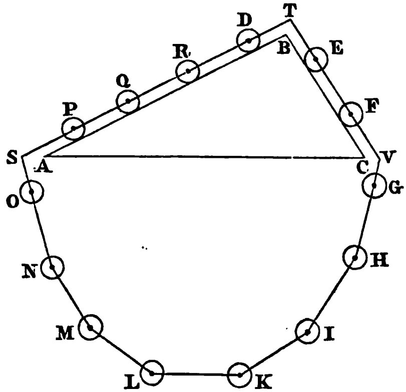
Let ABC be a triangle having its plane perpendicular to the horizon, and its base AC parallel to the same: and upon the side AB (which is double of BC), let there be placed a globe D, and upon BC another, E, equal to the former in weight and magnitude.

It is required to prove, that as AB : BC, *i.e.* as 2 : 1, so is the power of the weight E to that of D.

Let there be arranged round the triangle 14 globes, equal in weight and magnitude, and equidistant, strung upon a line passing through their centres, in such a manner that they may be able to turn about the said centres, and that there may be two globes upon the side BC, and four upon

* That is, as in page 74 of this Treatise.

the side AB; then as one line is to the other, so is the number of globes to the number of globes. Also at S, T, V let there be three fixed points, upon which the line or thread



may be able to run, and let the two portions above the triangle be parallel to the sides AB, BC; so that the whole may be able to turn freely and without catching upon the said sides AB, BC.

DEMONSTRATION.

If the power of the weights D, R, Q, P, be not equal to that of the two globes E, F, the one side will be more powerful than the other; let (if possible) the four D, R, Q, P be more powerful than the two E, F; but the four O, N, M, L are equal in effect to the four G, H, I, K; wherefore the eight globes D, R, Q, P, O, N, M, L, will be more powerful than the six, E, F, G, H, I, K; and since the stronger power must prevail over the weaker, the eight globes will descend, and the six will rise. Let this be so, and let D arrive at the place which is at present occupied by O, and so of the rest; thus E, F, G, H will assume the places occupied by P, Q, R, D, and I, K those occupied by E, F; notwithstanding the change therefore, the globes will have the same disposition as before, and for the same reason the eight globes will descend in virtue of their superior

gravity, and in descending will cause eight others to take their places, and so the motion will be perpetual; which is absurd. And the like demonstration will hold for the other side: the set of globes D, R, Q, P, O, N, M, L will therefore be in equilibrium with E, F, G, H, I, K. Now take away from the two sides those weights which are equal and similarly situated, as are the four globes O, N, M, L on the one side, and the four G, H, I, K on the other; the remaining four D, R, Q, P will be, and will continue to be, in equilibrium with the two E, F: wherefore E will have a power twice as great as that of D; as therefore the side AB to the side BC, or as $2 : 1$, so is the power of E to the power of D. Therefore, *If a triangle*, &c. Q. E. D.

Such is Stevinus's demonstration; from it he deduces the doctrine of the inclined plane in its most general form, that is in fact the laws of oblique forces. We shall not follow him any further; but the fundamental proposition just given is interesting from its ingenuity, and because it was the first independent proof of the laws of oblique forces.

EXAMINATION UPON CHAPTER VII.

1. The arms of a false balance are respectively 1 foot, and 1·05 in length; what will be the apparent value of 100 lbs. of tea, weighed out with such a balance (the tea being suspended from the longer arm), if the shop-keeper undertake to sell his tea at 4*s.* per pound?

2. If when a balance is suspended the beam be not horizontal, prove that if the want of horizontality arise from an inequality in the weight of the scale pans, the balance may be corrected by putting a weight into the lighter of the two, but that if it arise from a difference of length of the arms the balance cannot be so corrected.

3. Shew how to graduate the common steelyard.

4. Shew how to graduate the Danish steelyard, in which the fulcrum is moveable.

5. Find the relation of P to W in the single moveable pully.

6. In the first system of pullies.

7. In the second.

8. In the third.

9. Find the relation of P to W on the inclined plane.

10. In the screw.

11. What force is necessary to support a weight of 50 lbs. upon a plane inclined at an angle of 30^0 to the horizon, the force acting horizontally?

12. What force is necessary in the preceding problem, if the force act vertically?

13. When a given weight is sustained upon a given inclined plane by a force in a given direction, find the pressure upon the plane.

14. Given the weight, and the magnitude and direction of the sustaining force, find the inclination of the plane.

15. On an inclined plane the pressure, force, and weight, are as the numbers 4, 5, 7; find the inclination of the plane to the horizon, and the direction of the force.

16. What weight is that, which it would require the same exertion to lift as to sustain a weight of 4 lbs. upon a plane inclined at an angle of 30^0 to the horizon?

17. A weight W is sustained upon an inclined plane by a force P, acting by means of a wheel and axle, placed at the top, in such manner that the string attached to the weight is parallel to the plane. Given R and r the radii of the wheel and axle, find the inclination of the plane.

18. Two weights sustain each other upon two opposite inclined planes, by means of a string which is parallel to the planes; compare the pressures on the planes.

19. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 100 turns in the course of 12 inches, and which is acted upon by an arm 4 feet long?

20. Find the inclination to the horizon of the thread of a screw, which with a force of 5 lbs. acting at an arm of 2 feet, can support a weight of 300 lbs. on a cylinder of 2 inches radius.

If the length of the cylinder be 4 feet, find the entire length of the thread of the screw.

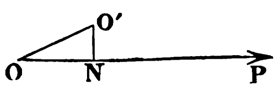
APPENDIX TO CHAPTER VII.

ON THE PRINCIPLE OF VIRTUAL VELOCITIES.

1. THE conditions of equilibrium of a system of particles, or a system of bodies, under the action of any forces, may be expressed in a very remarkable manner by means of a principle known as that of *Virtual Velocities.* This principle we shall here enunciate, and apply it to several cases of equilibrium; it is introduced here, because its chief interest to the reader of such an elementary treatise as this consists in the new view which it gives of the conditions of equilibrium of the various machines.

2. DEF. If we suppose a point at which any force acts to be very slightly displaced, and from the new position of the point a perpendicular to be dropped upon the direction of the force, then the line intercepted between the foot of this perpendicular and the original position of the point is called the *Virtual Velocity* of the point of application of the force, or sometimes more briefly the virtual velocity of the force.

Thus let O be the point at which the force P acts, and suppose it to be slightly displaced so as to be brought into the position O'; from O' draw the perpendicular $O'N$ on OP, then ON is the Virtual Velocity of P.



If the displacement of O is such that N falls between O and P, that is, if the virtual velocity is in the direction of the force, it is reckoned positive; if in the opposite direction, or N on the other side of O, it is reckoned negative.

3. It will appear from what has been said that the virtual velocity of a force is to a considerable extent an arbitrary quantity; and such is the fact, but it will be observed that when we have several forces acting at different points of a rigid body, the displacement of one point will in general determine the displacements of the others. For example, suppose we have two forces acting on the arms of a lever, then if we raise one extremity

of the lever through a small space, the other extremity is necessarily depressed through a space, the magnitude of which can be assigned.

4. If the displacement is made in the direction of the force, the whole displacement becomes, according to our definition, the virtual velocity, and if in a direction perpendicular to that of the force, the virtual velocity is zero. And in general we may regard the virtual velocity as the space through which the point of application is moved *in the direction of the force.* It will be seen also, since an arc and its sine are ultimately equal to each other, that when a force is acting perpendicularly to an arm of a lever, and the arm is made to turn through a very small angle, the small arc of a circle described by the point of application may be taken as the virtual velocity of the force.

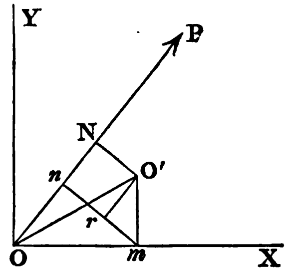
5. Hence we shall see something of the meaning of the term Virtual Velocity; for suppose we have any number of forces acting at different points, and that in consequence of an arbitrary motion of one of the points in the direction of the corresponding force through a very small space α, the other points of the system move in the directions of their respective forces through the spaces β, γ, &c.; then since these points move contemporaneously through the spaces $\alpha, \beta, \gamma \ldots$, these spaces measure the *rate* at which they respectively move; for example, suppose $\beta = 2\alpha$, $\gamma = 3\alpha$, &c., then the points must have moved at rates, or with velocities, which are in the ratio of 1, 2, 3, &c.; but these velocities are not real, since the parts of the system do not move in consequence of the forces which act upon them; if they did move, the question would be Dynamical, not Statical; hence the small spaces of which we have been speaking are called *Virtual* Velocities. And the student cannot too carefully bear in mind, that the motion which would seem to be implied by the term velocity is altogether of a geometrical character, that is, it is not due to the forces of the system, but is only a displacement supposed to be arbitrarily produced, without any reference to the nature of the forces necessary to produce it.

6. Having explained what is meant by virtual velocity, we shall be able to prove several propositions, which form particular cases of a very general principle known as that of Virtual Velocities, the proof of which we cannot give here, but of which it may be well to give the enunciation.

When a system of bodies is in equilibrium under the action of any forces, then if the system be very slightly displaced, the sum of the products of the forces and their respective virtual velocities will be equal to zero.

All that we shall do will be to prove this principle in those cases of equilibrium, which have been already considered, assuming the results which have been obtained.

7. *To prove the principle of virtual velocities in the case of a single particle, acted upon by any system of forces in the same plane.*



Let O be the particle, P any one of the forces, which makes an angle θ with a line OX drawn through O. Let the particle be displaced to O', and from O' draw $O'N$ perpendicular to OP, and let $ON = p$; also draw $O'm$ perpendicular to OX, and let $Om = x$, $O'm = y$; then it is easy to see, by drawing mn perpendicular to OP, and $O'r$ perpendicular to mn, that

$$p = On + O'r = x \cos\theta + y \sin\theta.$$

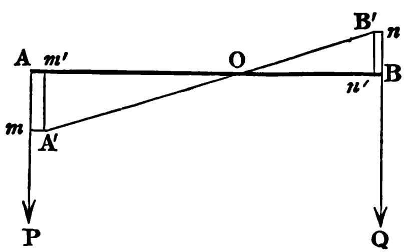
Similarly, if p', p''... be the virtual velocities of forces P', P''... acting at angles θ', θ''... with the line OX, we shall have

$$p' = x\cos\theta' + y\sin\theta',$$
$$p'' = x\cos\theta'' + y\sin\theta'',$$
$$\&c. = \&c.$$

$$\therefore Pp + P'p' + P''p'' + \ldots = x\,(P\cos\theta + P'\cos\theta' + P''\cos\theta'' + \ldots)$$
$$+ y\,(P\sin\theta + P'\sin\theta' + P''\sin\theta'' + \ldots) = 0,$$

by the general conditions of equilibrium established in Art. 10, page 56; which proves the principle of virtual velocities in this case.

8. *To prove the principle of virtual velocities in the case of the Lever.*



(1) Suppose the lever to be a straight lever AB, having arms $AO = a$, $BO = b$, and to be acted upon by forces P and Q perpendicular to the arms.

Let the lever be turned through a small angle about its fulcrum, so that the points A, B, are brought into the positions A', B', respectively; from A', B', draw $A'm$, $B'n$ perpendicular to the directions of the forces, and $A'm'$, $B'n'$, perpendiculars upon the lever. Then Am, Bn, or $A'm'$, $B'n'$ are the virtual velocities of P and Q.

Now we have seen, Art. 2, page 62, that

$$P \,.\, a = Q \,.\, b;$$

but by similar triangles $A'Om'$, $B'On'$,

$$\frac{A'm'}{a} = \frac{B'n'}{b};$$

$$\therefore P \,.\, A'm' = Q \,.\, B'n'.$$

Hence, not having regard to sign, we may say that

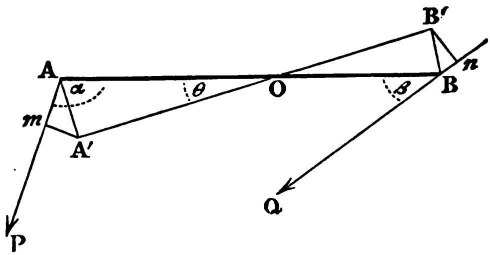
$P \times P$'s virtual velocity $= Q \times Q$'s virtual velocity.

Or if we denote $A'm'$ by p, and $B'n'$ by $-q$ (see Art. 2), we shall have

$$P \,.\, p + Q \,.\, q = 0,$$

which coincides with the general enunciation of the principle of virtual velocities given in Art. 6.

(2) Suppose the forces P and Q to act at any angles α and β with the lever AOB.



Let the lever be turned through a small angle as before, and call the angle θ. From A', B' the new positions of A and B draw $A'n$, $B'n$ perpendicular to the directions of the forces; then if θ be indefinitely small, Am, Bn will be the virtual velocities of P and Q. Join AA', BB'.

Then $Am = AA' \cos A'Am$

$$= 2a \sin\frac{\theta}{2} \cos\left(\alpha - 90^0 + \frac{\theta}{2}\right),$$

$$\left(\text{since } A'AO = 90^0 - \frac{\theta}{2}\right),$$

$$=2a \sin\frac{\theta}{2} \sin\left(\alpha+\frac{\theta}{2}\right);$$

similarly it will be found that $Bn=2b \sin\frac{\theta}{2} \sin\left(\beta-\frac{\theta}{2}\right)$,

$$\therefore \frac{Am}{Bn}=\frac{a \sin\left(\alpha+\frac{\theta}{2}\right)}{b \sin\left(\beta-\frac{\theta}{2}\right)}.$$

If we make θ indefinitely small, we shall have $\sin\left(\alpha+\frac{\theta}{2}\right)$ indefinitely nearly equal to $\sin \alpha$, and $\sin\left(\beta-\frac{\theta}{2}\right)$ to $\sin \beta$,

$$\text{and therefore } \frac{P\text{'s virtual velocity}}{Q\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots}=\frac{a \sin \alpha}{b \sin \beta}.$$

But we know, from Art. 2, page 62, that

$$\frac{P}{Q}=\frac{b \sin \beta}{a \sin \alpha},$$

$$\therefore P \times P\text{'s virtual velocity}=Q \times Q\text{'s virtual velocity},$$

or, having regard to the signs of the virtual velocities, and calling them p and $-q$,

$$P \,.\, p+Q \,.\, q=0,$$

as before.

This last demonstration is applicable to the case of any rigid body acted upon by two forces in the same plane, and having one point fixed; for through the fixed point we may draw a straight line intersecting the directions of the forces, and the points of intersection we may regard as the points of application of the forces. Hence in this general case the principle of virtual velocities is true.

9. *The Wheel and Axle.*

The condition of equilibrium being precisely the same as for the straight lever acted upon by two forces perpendicular to its arms, the demonstration will be the same as in that case.

10. *The Pully.*

In applying the principle of virtual velocities to pullies, we suppose the weight W to be raised through a small space, which small space will be its virtual velocity, and the corresponding space through which the point of application of P must be moved in order to keep the string stretched will be the virtual velocity of P.

(1) *The single moveable Pully.*

If in the figure, page 86, Art. 13, we suppose W raised through a small space a, the string on either side of the pully

will be shortened by the same quantity; consequently the point of application of P must be raised through $2a$, which will be P's virtual velocity.

But $$2P = W;$$
$$\therefore P \times 2a = W \times a,$$
or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

(2) *The first system of Pullies.*

In the figure of page 86, Art. 14, let W be raised through a small space a, then the lowest pulley rises through a space a, the second (reckoning from the lowest) through a space $2a$, the third through $2 \times 2a$ or $2^2 a$, and so on; hence the n^{th} pully will rise through a space $2^{n-1}a$, and the space through which P will descend will be $2^n a$.

But $$P \times 2^n = W;$$
$$\therefore P \times 2^n a = W \times a,$$
or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

(3) *The second system of Pullies.*

In the figure of page 87, Art. 15, let W be raised through a small space a; then if there be n strings between the two blocks, each of these will be shortened by a quantity a, consequently P will descend through a space na.

But $$P \times n = W;$$
$$\therefore P \times na = W \times a,$$
or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

(4) *The third system of Pullies.*

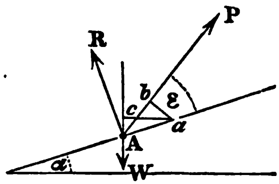
In the figure of page 88, Art. 16, let W be raised through a small space a; then the second pully (reckoning from the highest) will descend through a space a, and therefore the third pully will descend through $2a$; but in consequence of the rising of W, the third pully would have descended through a, even if the second had been fixed, therefore on the whole it descends through $2a + a$. In like manner the fourth descends through $2(2a + a) + a$, or $(2^2 + 2 + 1)a$; and the n^{th} through $(2^{n-2} + 2^{n-3} + \ldots + 1)a$, and P through

$$(2^{n-1} + 2^{n-2} + \ldots + 1)a, \text{ or through } (2^n - 1)a.$$

But $$P \times (2^n - 1) = W;$$
$$\therefore P \times (2^n - 1)a = W \times a,$$
or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

11. *The Inclined Plane.*

Let A be a particle, of weight W, which is kept at rest on an inclined plane by a force P, the direction of which makes an



angle ϵ with the plane; R the pressure of the plane on A; α the angle of the plane.

Suppose A to be moved along the plane to the point a; from a draw ab, ac perpendicular to the directions of P and W respectively; then Ab, Ac are the virtual velocities of P and W; R will have no virtual velocity, Art. 4.

Now $$Ab = Aa \cdot \cos \epsilon,$$
$$\text{and } Ac = Aa \cdot \sin \alpha;$$
$$\text{but } P \cos \epsilon = W \sin \alpha;$$
$$\therefore P \times Aa \cos \epsilon = W \times Aa \sin \alpha,$$
$$\text{or } P \times Ab = W \times Ac,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

12. *The Screw.*

It is evident that if the arm upon which the force P acts (see fig. page 94) be made to describe a complete revolution, the weight W will be raised or depressed through a space equal to the vertical distance between two threads of the screw; and the same proportion will be observed whatever be the actual maguitudes of the motions of P and W; consequently supposing these motions to be indefinitely small, we have (Art. 4)

$$\frac{P\text{'s virtual velocity}}{W\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots} = \frac{\text{circumference of circle described by } P}{\text{vertical distance between two threads}};$$

$$\text{but } \frac{P}{W} = \frac{\text{vertical distance between two threads}}{\text{circumference of circle described by } P};$$

$\therefore P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

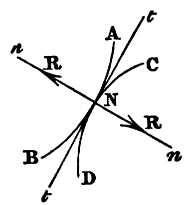
The student, after reading this article, will understand more clearly the reasoning concerning *Hunter's Screw*, given in the note on page 95.

13. We have thus proved the principle of virtual velocities in the case of all the simple machines. In any combination of these machines it is not difficult to conclude that the principle must also hold. A law which thus brings under one view the conditions of equilibrium in so many different cases will doubtless appear to the student one of great beauty and generality, although only a deduction from conditions previously established; but the principle of virtual velocities appears in its most striking light, when demonstrated in all its generality, and made the basis of mechanical investigations.

CHAPTER VIII.

ON FRICTION.

1. IN the preceding chapter we had occasion to speak of the effect of a smooth plane upon a body in contact with it, and we concluded that the effect would be to produce a force upon the body in the direction perpendicular to the plane. Upon the same principle we can conclude the nature of the force, which exists between any two smooth surfaces in contact. Let AB, CD be two smooth surfaces in contact at N; then they will exert upon each other a pressure, which (as we do not know its magnitude) we will call R. This pressure will be mutual; that is, if AB presses against CD with a force R, CD must of necessity press in the exactly opposite direction with an equal force: this is manifest from the nature of the case. The only question is, in what direction will the forces R act? Now since the surfaces are in contact at N, they must have a common tangent at that point; let it be tNt; draw nNn perpendicular to this tangent, then we call this line a *normal* to the surfaces, or the *common normal*. Since the surfaces are *smooth*, they cannot exert any action upon each other in the direction tNt, therefore the whole action must be in the direction nNn, or *in the direction of the common normal.*

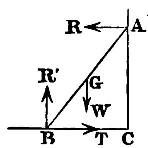


2. Although in practice no surface is perfectly smooth, yet the simplicity gained for mathematical investigations by the supposition of perfect smoothness is so great, that problems are usually solved with this imaginary condition. In the chapter upon machines, for instance, we have

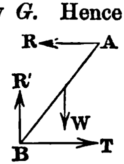
adopted this method. Consequently, it is necessary in solving problems to be able to determine at once the manner in which smooth surfaces act upon each other; and even if we take the actual case of bodies having some degree of roughness, it will be found that the nature of the mutual action in that case will be best understood by first becoming familiar with the simpler case of smooth surfaces.

3. One of the most common cases of mutual action is that of a particle, or a surface of indefinitely small magnitude, the end of a rod for instance, upon another surface, as that of a plane, a sphere, or the like. In this case it is to be observed, that the surface of finite magnitude determines the direction of the mutual pressure, because any line whatever may be considered as *normal* to the surface of a point. The meaning of this will be seen better from examples.

Let AB be a pole, a rod, or ladder, standing upon the smooth horizontal plane BC and resting against the smooth vertical wall AC; we must suppose the foot of the ladder restrained by a string BC, otherwise it will evidently slide down. Now, first, what will be the action of the wall upon the ladder at A? It will be a pressure of unknown magnitude, R, normal, that is perpendicular, to the wall.

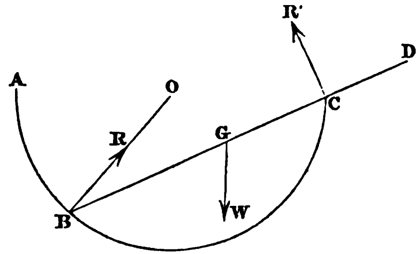


Again, what will be the action of the smooth horizontal plane at B? It will be a pressure of unknown magnitude, R', normal, that is perpendicular, to the horizontal plane. Let us complete the consideration of the forces acting upon AB. The string BC will produce a force, which we call the *tension* of the string, in the direction BC: this we will denote by T. And lastly, we have the weight of AB, which we may regard as one single force W, acting vertically at the centre of gravity G. Hence the problem is that of a rod AB, under the action of the forces represented in the accompanying figure; and we may divest ourselves of all notion of wall, ground, &c., and confine our attention to the system of forces as there represented.



Let us take another example: BD is a smooth heavy

rod or beam, resting (as represented in the figure) in a fixed smooth hemispherical bowl ABC, the centre of which



is O. There will be an action of the bowl upon the rod at B and C, which we will denote by R and R' respectively: what will be their directions?

At B, only the extremity of the rod, which we regard as a point, rests upon the surface of the bowl; consequently the pressure R will be perpendicular to the tangent at B, that is, its direction will pass through the centre of the sphere O. But at C the surface of the beam is in contact with the edge of the bowl, which we regard as indefinitely thin; consequently the pressure R' will be perpendicular to BD the direction of the beam.

In this case, as in the preceding, having once determined the directions of the various forces, and represented them, as in the figure, we dismiss the consideration of the bowl, and confine our attention to the equilibrium of the beam BD, acted upon by the three forces R, R', W.

The instances, which we have now considered, will be sufficient for our present purpose; the subject will be further illustrated in a subsequent chapter, in which we shall be employed in solving problems, and shall shew by what conditions the unknown pressures, of which we have been speaking, must be determined.

4. We now pass to the case of *rough* surfaces. When a force tends to draw one body over the surface of another, there is, as we know by constant experience, a resistance to motion; and this resistance we call the force of *friction*. Its intensity manifestly depends partly upon the nature of the surfaces in contact; thus it is more easy to drag a

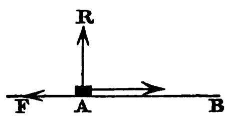
load over a wooden floor than over a turnpike road, still more easy over a polished marble floor, and the resistance of this last may again be diminished by throwing upon it a small quantity of oil. Now, inasmuch as in all practical problems the force of friction enters in a very important manner, it has been the business of scientific men to determine as far as possible by experiment what are the laws of its action. The following are the conclusions to which they have been led.

5. Suppose two bodies to be in contact, and one of them to be on the point of sliding over the surface of the other in consequence of some force acting upon it, and suppose it to be restrained from motion by the force of friction, then

(1) *The force of friction is proportional to the mutual pressure between the two bodies; and*

(2) *The force is independent of the extent of the surface in contact.*

Suppose, for instance, that a body A rests upon a plane surface, and that the pressure exerted upon the surface by its own weight or otherwise is R; and suppose it to be on the point of moving in the direction AB under the action of any force, and to be restrained from moving by the opposite force of friction F. Then F is proportional to R, or $F \propto R$; that is, if we double R, we double F; a weight of 2 lbs. will offer twice as much resistance to motion over a rough horizontal plane as a weight of 1 lb.; a weight of 3 lbs., three times as much, and so on.



This relation is generally expressed by the equation

$$F = \mu R,$$

where μ is a quantity called the *coefficient of friction*, and depending upon the nature of the surfaces in contact; thus, for wood it will have one value, for slate another, ivory another. It will also depend upon the degree of polish which the surfaces may have, the absence or presence of any oily matter between the bodies, and so on.

The second law teaches us, that provided the pressure is the same the friction does not depend upon the extent of surface in contact; that is, a weight of 1 lb. will not exert

more friction if it rests upon a surface of two square inches than if it rests on a surface of one*. This law however is not true in extreme cases, as when the surface in contact is reduced very nearly to a point.

On the whole, the formula $F=\mu R$ will express all the laws of friction, if we remember that μ is a quantity independent of R, independent of the extent of surface in contact, and dependent only upon the nature of the surfaces.

6. It will be remembered, that the friction of which we here speak is that friction which exists when the two surfaces with which we are concerned are *on the point* of sliding one upon the other; this is called *Statical Friction.* When motion actually takes place the amount of friction is in general different; this is called *Dynamical Friction,* and will not occupy our attention at present; we may however mention in passing, that the same two laws hold for Dynamical as for Statical friction, but that there is in addition this law, that the friction is independent of the velocity with which the surfaces move.

The following are a few results deduced by experiment:

Wood upon wood, $\mu = \cdot 50$.

Wood upon metal, $\mu = \cdot 60$.

Metal upon metal, $\mu = \cdot 18$.

These values may be much reduced by introducing between the surfaces any oily substance; for instance, in the last case, if olive oil be introduced between the metals, $\mu = \cdot 12$.

7. Forasmuch as friction depends entirely upon the mutual normal pressure, the consideration of friction does not introduce any new unknown force into a problem. And in general, when a problem can be solved upon the supposition of the surfaces involved in it being smooth, it can

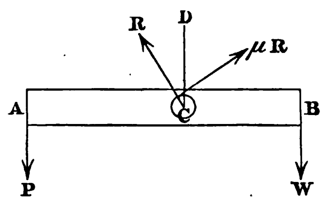
* This result is not so strange as it may appear at first sight. For suppose that a body of a certain weight rests upon another with a certain extent of surface in contact, and suppose that without altering the weight we diminish the surface in contact by one half, then if we regard the pressure as uniformly distributed over all the points of the surface in contact, we shall have diminished the number of points by one half, but we shall have *doubled the pressure* upon each point, since upon the whole the same pressure has to be sustained as before. And since the friction varies directly as the pressure, the *friction* at each point will be *doubled.* Hence, on the whole, as we diminish the extent of surface in contact, we increase the intensity of the friction, and therefore the friction will be *cæt. par.* independent of the extent of surface.

also be solved with a little increase of trouble, but with no real increase of difficulty, upon the supposition of the surfaces being rough; at least, it can be solved *for the state bordering on motion*. In the machines, for instance, of the preceding chapter, friction is practically a very important force, and one which may by no means be omitted; and it will be found, after what has been said, that there is no difficulty in introducing the consideration of it, provided only that we suppose P to be on the point of descending, or on the point of ascending; these will form two limiting cases, between which all other possible cases will be included.

8. We will now give a few examples of the solution of problems with friction.

Ex. 1. The lever with friction.

Let AB be a lever working upon a cylindrical axis C; then if we suppose friction to exist between the surface of the axis and the aperture in which it works, and suppose that the lever is horizontal, but the extremity B on the point of descending, the pressure upon the axis will not be in the vertical direction CD, but inclined to it at some angle θ, (suppose). Call this pressure R, then we shall have a force of friction μR, (as in the figure), tangential to the cylindrical axis, that is, perpendicular to the direction of R.



Let $AC=a$, $BC=b$, and let r be the radius of the cylindrical axis; then, by resolving horizontally and vertically and taking moments about C, we shall have the following three equations;

$$R \sin \theta - \mu R \cos \theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$R \cos \theta + \mu R \sin \theta = P + W \ldots\ldots\ldots(2),$$

$$Pa = Wb + \mu Rr \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Equation (1) gives us,

$$\tan \theta = \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and from (2) and (3) we have,

$$\mu r(P+W) = (\cos \theta + \mu \sin \theta)\mu Rr,$$

$$= (\cos \theta + \mu \sin \theta)(Pa - Wb);$$

$$\therefore P\{a(\cos \theta + \mu \sin \theta) - \mu r\} = W\{b(\cos \theta + \mu \sin \theta) + \mu r\},$$

$$\text{or } \frac{P}{W} = \frac{b(\cos \theta + \mu \sin \theta) + \mu r}{a(\cos \theta + \mu \sin \theta) - \mu r}.$$

If we put for μ its value from equation (4), we have,

$$\frac{P}{W}=\frac{b(\cos\theta+\sin\theta\tan\theta)+r\tan\theta}{a(\cos\theta+\sin\theta\tan\theta)-r\tan\theta}$$
$$=\frac{b+r\sin\theta}{a-r\sin\theta}\ \dots\dots\dots\dots\dots\dots\dots\dots(5).$$

In applying this formula we may find θ from equation (4) by means of a trigonometrical table, and then put the value of $\sin\theta$ in (5). And it may be remarked, that the equations of problems, in which friction is involved, generally assume their simplest form when we represent the coefficient of friction by the tangent of a certain angle. On the other hand, we may, if we please, express the ratio of P to W in terms of μ, and we have then

$$\frac{P}{W}=\frac{b+r\dfrac{\mu}{\sqrt{1+\mu^2}}}{a-r\dfrac{\mu}{\sqrt{1+\mu^2}}}.$$

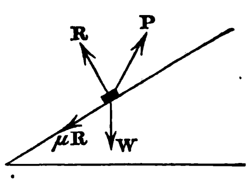
We have supposed that P is on the point of descending; if on the contrary, W be on the point of descending, we should find in like manner that

$$\frac{P}{W}=\frac{b-r\dfrac{\mu}{\sqrt{1+\mu^2}}}{a+r\dfrac{\mu}{\sqrt{1+\mu^2}}},$$

a result which may be obtained from the other *by changing the algebraical sign of* μ.

Ex. 2. The inclined plane with friction.

We will take the most general case, in which P acts in a direction making any angle ϵ with the inclined plane; and we will suppose the weight W to be on the point of *ascending;* then the forces will be such as represented in the figure. If a be the angle of the plane, we shall have by resolving parallel and perpendicular to the plane,



$$P\cos\epsilon-W\sin a-\mu R=0\dots\dots\dots\dots\dots(1),$$
$$P\sin\epsilon-W\cos a+R=0\dots\dots\dots\dots\dots(2).$$

Multiplying equation (2) by μ and adding it to (1) there results

$$P(\cos\epsilon+\mu\sin\epsilon)=W(\sin a+\mu\cos a),$$
$$\therefore\ \frac{P}{W}=\frac{\sin a+\mu\cos a}{\cos\epsilon+\mu\sin\epsilon}.\ \dots\dots\dots\dots\dots(3).$$

If W be on the point of descending, the force of friction will be in the opposite direction, and the result will be

$$\frac{P}{W} = \frac{\sin \alpha - \mu \cos \alpha}{\cos \epsilon - \mu \sin \epsilon} \text{.................(4).}$$

If, according to the remark made in the last example, we put $\tan \beta$ instead of μ, where β is a *subsidiary angle* determined by the equation

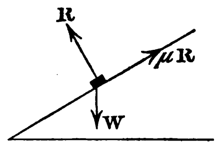
$$\tan \beta = \mu,$$

then (3) and (4) become respectively,

$$\frac{P}{W} = \frac{\sin (\alpha + \beta)}{\cos (\epsilon - \beta)}, \text{ and } \frac{P}{W} = \frac{\sin (\alpha - \beta)}{\cos (\epsilon + \beta)}.$$

By making $\beta = 0$ we reduce these expressions to those already investigated in page 91 for the case of the smooth inclined plane.

Ex. 3. If in the last result we make $P=0$, we have $\alpha = \beta$, or the subsidiary angle β is the inclination of the plane for which a weight will just not slide. Let us, as another example, determine directly the greatest angle at which a plane may be inclined without allowing a weight placed upon it to descend.



The forces will be as in the figure; resolving horizontally and vertically, we have

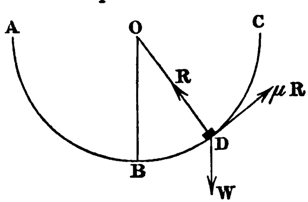
$$R \sin \alpha - \mu R \cos \alpha = 0 \text{............(1),}$$
$$R \cos \alpha + \mu R \sin \alpha = W \text{ ....(2);}$$

from (1) there results, $\tan \alpha = \mu$,

which gives us the angle required. Equation (2) gives us the value of R.

Ex. 4. If the interior surface of a hemispherical bowl be perfectly smooth, it is manifest that a particle cannot rest except at the lowest point of the bowl; but if the surface be rough, the particle may rest at a distance from the lowest point: let us determine the limits within which this is possible.

Let ABC be the bowl, O its centre, OB vertical, D the furthest point from B at which a weight W will rest, $BOD = \theta$. The forces will be as in the figure. Then, resolving horizontally, we shall have



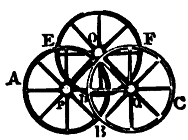
$$R \sin \theta - \mu R \cos \theta = 0,$$
$$\text{or } \tan \theta = \mu,$$

which equation determines the position of D.

9. These examples will be sufficient to illustrate in general the method of treating problems, when the surfaces involved in them are not perfectly smooth; and they will shew that no new unknown forces are introduced into problems by the consideration of friction, provided we take only the extreme case in which the surfaces in contact are on the point of sliding one upon another.

10. It may be observed that in practice friction may be regarded as a most useful force; the climbing a hill, or even walking upon level ground, would be impossible without it; for in this latter case the pressure of the foot upon the ground must, unless for this action of friction, be accurately normal to the horizon, a condition which in practice could not be satisfied. To take another instance; when a nail is driven into a piece of wood, it would not remain in its place if it were not for the force of friction. In these and in hundreds of every-day instances, friction is a most useful force. Then, on the other hand, in the construction of machines friction is often far from useful, one great difficulty with which mechanicians have to contend being in some cases that of doing away with the effects of friction; the construction of clocks and watches is a good example.

11. Many devices are adopted for the purpose of diminishing the resistance due to friction, in cases in which its action interferes with work to be done. Thus a large block of stone is transported by placing rollers underneath it, the intention beiug to diminish the enormous amount of friction which would exist between the heavy block and the ground. The wheels of carriages are the same device in a more delicate form; in this case the friction is reduced to that of the polished surface of the axle, which by a constant supply of oil or grease is made very small. And there is the very ingenious device of *friction wheels*. To understand this invention, let AB, CD be two wheels, turning about horizontal axes P, Q; and let there be upon the same axle Q a third wheel (not seen in the figure) behind CD and exactly similar to it; lastly, let EF be a wheel which we

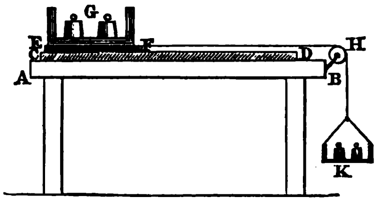


desire to cause to turn with as little friction as possible; then its axle O, instead of running in a socket, is allowed to rest upon the surfaces of the three friction wheels just

described. The consequence of this arrangement is, that any tendency to friction between the axle O and the surface of the friction wheels, instead of being resisted by a fixed surface, causes the friction wheels to revolve, and thus the amount of friction is diminished to a very remarkable extent.

12. The following is a description of a simple apparatus, which may be conveniently used for the purpose of investigating the laws of friction.

AB is a firm horizontal table, upon which can be

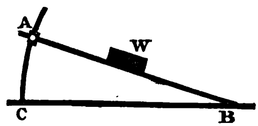


fastened a flat piece *CD* of one of the substances concerning which we desire to make experiments; *EF* is the other substance, which lies upon the surface of *CD*, and bears upon it the case *G* which may be weighted to any extent. A horizontal string *FH*, attached to *EF*, passes over a pully at *H* and carries a scale *K* which also may be weighted as we please. In order to make experiments upon the friction between two given substances *CD* and *EF*, we have only to put a certain weight in *G*, and load the scale *K* until *EF* just begins to move; then it will be seen that the weight of the case *G* and its contents, added to the weight of *EF*, measures the pressure between the substances, and the weight of the scale *K* and its contents measures the friction for the state bordering on motion. It will be easy to vary the amount of the weight in *G*, the nature of the surfaces in contact, and the extent of the surfaces; the result of experiments so made is to establish the laws of friction which have been given in this chapter.

It should be remarked, that in making experiments with soft bodies, such as wood for instance, the amount of friction will depend to a certain extent upon the time during which the surfaces have been in contact, being less at first than it is afterwards. This is quite what we should expect; the increase of friction only takes place during a

limited time: thus with wood upon wood, for instance, the friction attains it greatest value after about two or three minutes; for wood upon metals a longer time is required for the friction to attain its permanent value, sometimes even as much as several days.

13. By the apparatus described in the preceding article we might of course determine the value of μ for any given substances. The value may however be obtained perhaps more simply thus. It has been shewn that if a be the greatest angle of inclination of a plane of given substance upon which a portion of another given substance can rest, and if μ be the coefficient of friction for the two substances, then $\tan a = \mu$. Hence to determine μ it will be sufficient to determine a; suppose now that AB is a bar moving about a hinge B in a horizontal bar BC, and that CA is a graduated circular rim; suppose also that upon AB we can screw a plate of one of the substances, and upon this plate let the other substance W be placed; then let the extremity A be raised until W is on the point of descending; ABC will be then the angle a, and will thus give us the value of μ.



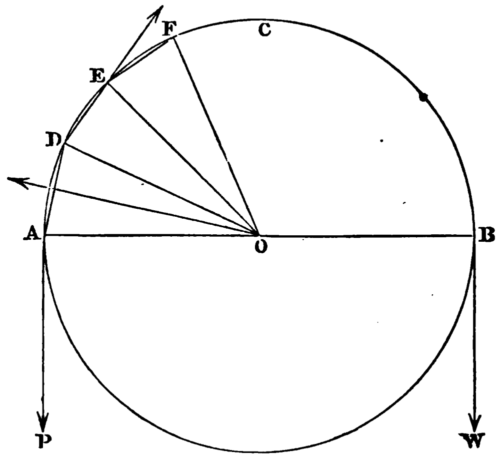
14. We will conclude this chapter by considering a problem of some interest, which involves the principles of friction. Every one must have observed the great advantage gained by a sailor, who gives a rope two or three twists upon a post, and is able by so doing to maintain his hold upon the rope when a very great force is applied at the other end. The advantage gained manifestly depends upon the friction between the rope and post, and may be made intelligible as follows.

Let us consider this problem: Two weights, P and W, are connected by a string which passes over a given rough horizontal cylinder: P is on the point of descending, required the ratio of P to W.

Let ACB be a section of the cylinder; O its centre. Then the string connecting P and W will be in contact with the cylinder throughout the semicircumference ACB.

Now the difficulty of the problem consists in this, that the direction of the string upon the cylinder is constantly changing its direction from point to point: in order to

overcome this difficulty let us cut up the arc ACB into n small equal parts, AD, DE, EF, ... ; join AD, DE, EF, ... and, instead of considering the circular arc ACB, let



us consider the polygon $ADEF$...... and let us suppose the string to rest upon a prismatic post of many sides instead of a cylindrical post.

Let us consider how the piece of string AD is held in equilibrium; there will be the force P at A, and a force at D in the direction of DE, which will be composed partly of the tension of the string, which we will call P_1, and partly of the friction, which we will call F. There will be a third force, or rather a system of forces perpendicular to AD arising from the pressure of the surface of the post upon the string, but these forces we may consider to be equivalent to one single force acting at the middle point of AD perpendicular to it and proportional to its length; this force we will call therefore $p \,.\, AD$. Lastly, it will be observed that the angle AOD is by construction equal to the n^{th} part of two right angles, or $\dfrac{\pi}{n}$.

Now for the equilibrium of AD we have, resolving parallel and perpendicular to it

$$P = P_1 + F \text{......................................} (1),$$

$$p \,.\, AD = P \cos DAO + (P_1 + F) \cos ADO \text{......} (2),$$

and by the law of friction

$$F = \mu \,.\, p \,.\, AD \text{......................................} (3).$$

Now $\cos DAO = \cos ADO = \sin \frac{AOD}{2} = \sin \frac{\pi}{2n}$,

$$\text{also } AD = AO \text{ chd.} AOD = 2AO \sin \frac{\pi}{2n};$$

$\therefore$ equations (1), and (2), give us,

$$2pAO \sin \frac{\pi}{2n} = 2P \sin \frac{\pi}{2n};$$

$$\therefore p = \frac{P}{AO};$$

$$\therefore \text{ from (3), } F = \mu P \,.\, \frac{AD}{AO} = 2\mu P \sin \frac{\pi}{2n};$$

and therefore from (1),

$$P_1 = P - 2\mu P \sin \frac{\pi}{2n} = P \left\{1 - 2\mu \sin \frac{\pi}{2n}\right\}.$$

In like manner if P_2 be the tension of the string at E, we have

$$P_2 = P_1 \left\{1 - 2\mu \sin \frac{\pi}{2n}\right\}$$

$$= P \left\{1 - 2\mu \sin \frac{\pi}{2n}\right\}^2.$$

And so on. But we have divided the circumference ACB into n parts; consequently, according to our notation,

$$W = P_n = P \left\{1 - 2\mu \sin \frac{\pi}{2n}\right\}^n.$$

This formula will explain the advantage of wrapping a cord upon the surface of a many-sided prismatic post, from which the case of a cylinder is not a very difficult step. For, suppose we have a prismatic post of $2n$ sides; and let P and Q be the two forces which are in equilibrium at the extremities of a rope wrapped upon it, P being on the point of overcoming Q; then it will appear, that if we make for shortness' sake $\left(1 - 2\mu \sin \frac{\pi}{2n}\right)^{2n} = r$,

for one complete turn upon the post, $Q = Pr$,
... two .... $Q = Pr^2$,
... three .... $Q = Pr^3$,

and so on: that is to say, as the number of turns increases in an *arithmetical* progression the mechanical advantage increases in a *geometrical*.

The same conclusion will hold for a cylindrical surface. In this case the quantity which has been called r has a value, which might be obtained by making the quantity n which occurs in it indefinitely great*. But without doing this it may be remarked that what is true for a polygon of as many sides as we please may be concluded to be true of a circle; and therefore it may be concluded, that if we have a cylindrical post round which a rope is wrapped, the mechanical advantage thereby gained increases in a geometrical progression as the number of turns of the rope increases in an arithmetical. Thus, if with the exertion of a force of 6 lbs. I am able with one turn of the cord to sustain a pressure of 24 lbs., then $r = \frac{6}{24} = \frac{1}{4}$, and two turns of the rope will enable me to sustain 6×4^2 or 96 lbs., three turns 6×4^3 or 384 lbs., and so on.

EXAMINATION UPON CHAPTER VIII.

1. Determine the direction in which the mutual pressure of two smooth surfaces in contact takes place.

2. Define friction, and enunciate its laws as determined by experiment.

3. Investigate the relation of P to W in the case of the lever, taking into account the action of friction.

4. The same for the inclined plane.

5. The same for the screw.

6. Find the greatest angle of elevation of an inclined plane for which it is possible for a lump of a given substance to rest upon it.

* The result would in fact be,

$$W = Pe^{2m\mu\pi},$$

where e is the base of Napier's logarithms, and m is the number of times the cord is wrapped round the cylinder.

7. A heavy particle is placed upon the exterior of a rough sphere; find the limits within which equilibrium is possible.

8. Given the magnitude of the horizontal force (P), which will just support a given weight (W) upon a plane of given inclination (a); determine the coefficient of friction.

9. There is a block of wood which can be just lifted by the combined strength of two men, determine the greatest angle of elevation of a wooden inclined plane upon which it can be supported by one man, ($\mu = \frac{1}{3}$); the man exerting a force parallel to the plane.

10. Prove that in the preceding problem the elevation of the plane, upon which one man can support the weight, is twice as great as that for which the weight would rest by itself.

11. Explain the manner of making experiments concerning the laws of friction.

12. Assuming the law, that when the extent of surface in contact of one body resting on another is given, the friction for the state bordering on motion varies as the pressure; deduce the truth of the other law of friction, namely, that the friction is independent of the extent of surface in contact.

13. What is meant by *friction wheels?* explain their use.

CHAPTER IX.

PROBLEMS.

IN this concluding chapter we shall give a number of problems, of such a nature as to be capable of solution by means of the principles already laid down. Some of the problems will be solved by way of illustration; in some hints towards solution will be given; and some will be left entirely to the ingenuity of the student.

The following short collection of hints and rules for the general method of treating problems may probably be found useful.

1. Draw a figure of the system as accurately as possible, representing by arrows the directions of the various forces; all forces, of which the magnitude is not known, to be denoted by symbols P, Q, R, &c.

2. If the system contains more than one body, the action and reaction between the various bodies at their points of contact must be considered; the action and reaction between two bodies will be always equal in magnitude and opposite in direction, and if the bodies be smooth the action and reaction will take place in the line of the common normal at the point of contact. When these forces have been taken into account, the equilibrium of each component body of the system may be considered separately.

3. Resolve the forces acting upon each body of the system in two directions at right angles to each other, and equate each result to zero: take moments about any point for each body, and equate the result to zero.

In the resolution of the forces, it is not generally of any serious consequence what directions of resolution are chosen; but in taking moments it is desirable to choose the point about which they are estimated, in such a manner as to give the simplest results; thus, if the directions of two or more forces pass through a point, it is generally desirable to take the moments with reference to that point.

4. Count all the mechanical equations thus produced, and count the unknown quantities both mechanical and geometrical; if the number of unknown quantities exceed that of the equations, the deficiency must be supplied by geometrical equations, that is, by equations expressing necessary geometrical relations amongst the parts of the system.

5. If the surfaces of bodies which act upon each other be rough, the solution of a problem is in general indeterminate except for the state bordering on motion, that is, for the position in which the bodies are on the point of sliding upon each other. In this limiting case, the friction acts in the direction opposite to that in which motion would take place, and is proportional to the mutual normal pressure.

6. The process which has been here described may not unfrequently be abbreviated by artifices peculiar to individual problems. Especially it may be noticed, that when there are only three forces acting upon a body, the position of equilibrium may generally be simply investigated by making the construction of the figure satisfy the necessary condition of the directions of three forces all passing through the same point.

7. The student must not be surprised if the solution of a problem lead to an equation, either algebraical or trigonometrical, which he is unable to solve; indeed it not unfrequently happens, that a problem of apparent mechanical simplicity leads to such a result. This usually arises from the fact of the mode of solution adopted necessarily involving other cognate problems, and therefore giving the solutions of them as well as of that with which he is engaged.

8. It may be remarked, as a matter worthy of the student's attention, that the thorough comprehension of

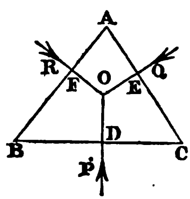
one problem, solved by himself, will do more to assist him in understanding the principles of Statics than the half-comprehension of many. On this account it is recommended to him, when he has solved a problem in one way, to vary his method of proceeding and solve the problem again. Thus; he may assume new directions of resolution, and a new point with respect to which to take moments; or he may endeavour to abbreviate the solution by geometrical construction; or he may vary the circumstances of the problem in some of its details. Examples of this will be found in the collection of problems following.

9. In making use of the collection of problems, the student is recommended to consider well, and to attempt to solve, those problems of which the solution is given in full, before he examines the solutions. The solved problems will be, comparatively speaking, of little profit to him, unless used with this caution.

With these hints the student may enter upon the consideration of the problems which follow, and which of course might be indefinitely multiplied. They will be found to be not exclusively problems concerning the position and conditions of equilibrium of rigid bodies; amongst them are problems concerning the equilibrium of a particle and the properties of the centre of gravity, and some concerning the equilibrium of a string.

I. Three forces act upon the sides of a triangular board, and in directions perpendicular to the sides: to prove that for equilibrium the forces must be proportional to the sides upon which they act.

Let ABC be the triangle; P, Q, R the forces acting at the points D, E, F respectively; produce the directions of P and Q to meet in O, then FO must be the direction of R, otherwise there could not be equilibrium.



Then by a theorem already proved, p. 56,

$$P : Q : R :: \sin EOF : \sin FOD : \sin DOE,$$
$$:: \sin A : \sin B : \sin C,$$
$$:: BC : CA : AB.$$

II. Treat the preceding problem by resolving the forces and taking moments. Shew that if $BD=x$, $CE=y$, $AF=z$, this method of solution leads to the conclusion,

$$x \sin A + y \sin B + z \sin C = \frac{a^2+b^2+c^2}{abc} \times \text{area of triangle};$$

and explain the meaning of this condition.

III. If θ be the angular distance of a body from the lowest point of a circular arc in a vertical plane, the force of gravity in the direction of the arc : that in the direction of the chord :: $2\cos\frac{\theta}{2} : 1$.

IV. A straight lever of uniform thickness, the length and weight of which are given, has two weights P and Q attached to its extremities, and is kept in equilibrium partly by a fulcrum at a given point, and partly by another fulcrum on which it presses with a given force; required the position of this latter fulcrum.

V. A beam, 30 feet long, balances about a point at one-third of its length from the thicker end; but when a weight of 10 lbs. is suspended from the smaller end, the fulcrum must be moved 2 feet towards it in order to maintain equilibrium. Find the weight of the beam.

VI. A heavy body is to be conveyed to the top of a rough inclined plane, the angle of inclination being α; prove that if the coefficient of friction be greater than $\dfrac{\sin\left(45^\circ - \frac{\alpha}{2}\right)}{\sin\left(45^\circ + \frac{\alpha}{2}\right)}$, it will be easier to lift the body than to drag it up by means of a cord parallel to the plane.

VII. If G be the centre of gravity of the triangle ABC, then

$$AB^2 + AC^2 + BC^2 = 3\{GA^2 + GB^2 + GC^2\}.$$

VIII. CA, CB are the arms of a bent lever; G is the centre of gravity of the lever; prove that

$$CG^2 = (CA - CB)^2 + \frac{4CA^2 . CB^2}{(CA+CB)^2} \cos^2 \frac{ACB}{2}.$$

IX. Two weights P and Q, connected by a string of given length, balance each other upon the surface of a sphere. Required the position of equilibrium.

X. If three weights, placed at the angular points A, B, C of a triangle, are respectively proportional to the opposite sides

a, b, c; prove that the centre of gravity of the weights is a point, the distances of which from A, B, C, are respectively,

$$\frac{2bc}{a+b+c}\cos\frac{A}{2},\ \frac{2ca}{a+b+c}\cos\frac{B}{2},\ \frac{2ab}{a+b+c}\cos\frac{C}{2}.$$

XI. Two forces F and F', acting in the diagonals of a parallelogram, keep it at rest in such a position that one of its edges is horizontal; shew that $F\sec\alpha = F'\sec\alpha' = W\cos(\alpha+\alpha')$, where W is the weight of the parallelogram, α and α' the angles between its diagonals and the horizontal side.

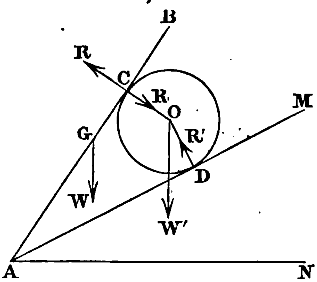
XII. A uniform beam AB, of given length and weight, rests with one end on a given inclined plane, and the other attached to a string AFP passing over a pully F at a given point of the inclined plane. Kuowing the weight P fixed to the other end of the string, find the position in which the beam rests.

XIII. Find the limits of possibility of the preceding problem.

XIV. Also solve the problem for the particular case in which the string AF is horizontal.

XV. By means of tables obtain a numerical result for the angle between the beam and the plane, when the inclination of the plane is 45^0, the weight of the beam 4 lbs., and $P = 3$ lbs.

XVI. A smooth sphere of which the centre is O, rests upon an inclined plane AM at D, and is kept in equilibrium by the uniform beam AB, which, moving about a hinge at A, presses upon the sphere at C. Required the conditions of equilibrium.



Let $\alpha = MAN$, the angle of the plane; W, W' be the weights of the beam and sphere respectively. At the point C there will be an action of the sphere upon the beam, and an equal and opposite action of the beam upon the sphere; call this R, its direction will be perpendicular to the beam and through the centre of the sphere. At D there will be an action of the plane upon the sphere, R' suppose; its direction will be DO. The beam will be kept in equilibrium by the forces R, W acting at G the middle point of AB, and a pressure at A which we need not consider; the sphere, by the three forces W' at O, R and R'.

Let r = the radius of the sphere; $AC = AD = x$; $CAD = \theta$.

Then for the beam, taking moments about A,

$$Rx = Wa\cos(\theta+a) \ldots\ldots\ldots\ldots (1).$$

For the sphere,

$$R\sin(\theta+a) = R'\sin a \ldots\ldots\ldots\ldots (2),$$

$$R\cos(\theta+a) + W' = R'\cos a \ldots\ldots\ldots\ldots (3).$$

And we have the geometrical relation,

$$a = x\tan\frac{\theta}{2}, \ldots\ldots\ldots\ldots (4).$$

Multiplying (2) by $\cos a$ and (3) by $\sin a$ and subtracting, we have,

$$R\{\sin(\theta+a)\cos a - \cos(\theta+a)\sin a\} - W'\sin a = 0,$$

$$\text{or } R\sin\theta = W'\sin a;$$

$\therefore$ combining this equation with (1), there results,

$$\frac{x}{\sin\theta} = \frac{Wa\cos(\theta+a)}{W'\sin a},$$

$$\text{or } \frac{W\cos(\theta+a)}{W'\sin a} = \frac{\cot\frac{\theta}{2}}{\sin\theta} = \frac{\cot\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1}{2\sin^2\frac{\theta}{2}};$$

$$\therefore 2\frac{W}{W'}\cos(\theta+a)\sin^2\frac{\theta}{2} = \sin a,$$

$$\text{or } \sin a\,\operatorname{cosec}^2\frac{\theta}{2} = (2\cos a\cos\theta - 2\sin a\sin\theta)\frac{W}{W'},$$

$$\frac{W'}{W}\left(1+\cot^2\frac{\theta}{2}\right) = 2\cot a\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) - 4\sin\frac{\theta}{2}\cos\frac{\theta}{2};$$

$$\therefore \frac{W'}{W}\left(1+\cot^2\frac{\theta}{2}\right)^2 = 2\cot a\left(\cot^2\frac{\theta}{2} - 1\right) - 4\cot\frac{\theta}{2},$$

$$\text{or } \frac{W'}{W}\left(\frac{x^2}{a^2}+1\right)^2 + 4\frac{x}{a} = 2\cot a\left(\frac{x^2}{a^2} - 1\right).$$

This is the simplest form to which the equation can be reduced; it has been already remarked, that not unfrequently statical problems of no great degree of complication lead (as in this case) to equations which we are not able to solve.

If the position of equilibrium be assigned, that is, if θ or x be given, the equation which we have obtained gives us the ratio of the weight of the sphere to that of the beam, for we have

$$\frac{W'}{W} = \frac{2\cot a\,(x^2-a^2) - 4ax}{(x^2+a^2)^2}.$$

From this equation we can obtain a limit of the possibility of the problem; for $\frac{W'}{W}$ must be positive, therefore we must have

$$2\cot a\,(x^2-a^2) - 4ax \text{ positive,}$$

$$\text{or } x^2 - a^2 > 2ax \tan \alpha,$$

$$\text{or } \tan \alpha < \frac{x^2 - a^2}{2ax}.$$

Suppose, for instance, that $x = 2a$, then

$$\tan \alpha \text{ must be} < \tfrac{3}{4} < .75,$$

$$\text{or } \alpha < 36^0\, 52';$$

$$\text{and then } \frac{W'}{W} = \frac{2 \cot \alpha \times 3 - 8}{25}.$$

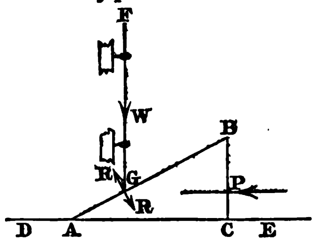
Thus if $\alpha = 30^0$, $\cot \alpha = \sqrt{3}$,

$$\text{and } \frac{W'}{W} = \frac{6\sqrt{3} - 8}{25} = \frac{10.39 - 8}{25} = \frac{1}{10} \text{ approximately};$$

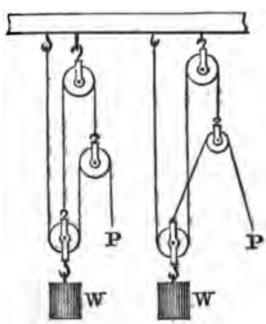
that is, if the inclination of the plane be 30^0, and the distance of either point of contact of the sphere from the hinge equal to the diameter, then for equilibrium the beam must be about 10 times as heavy as the sphere.

In the particular case of the weights of the sphere and beam being equal, the biquadratic upon which the problem depends is reducible to a cubic: this the student may prove.

XVII. *ABC* is a piece of wood, in the form of a right-angled triangle, resting upon a smooth horizontal plane *DE*. The heavy vertical beam *FG*, whose weight is *W*, passing through two rings rests upon the inclined plane *AB*. Find the horizontal force *P*, which is necessary to prevent motion.

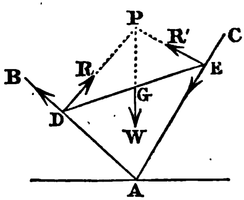


XVIII. Find the relation of *P* to *W* in each of the accompanying systems of Pullies.



XIX. A uniform rod, 50 feet long, is cut into two pieces of 18 and 32 feet respectively, and these are placed upon a smooth horizontal plane with their lower extremities pressing against each other, and the upper extremities leaning against two smooth vertical planes; also the two rods are at right angles to each other; find the distance between the walls.

XX. A uniform beam rests with its extremities upon two given inclined planes; to find the position of equilibrium.



Let AB, AC be the inclined planes, α, β, their respective inclinations; DE the beam, G its middle point or centre of gravity, W its weight, $2a$ its length; R, R' the actions of the two planes at D and E, which will be perpendicular to AB and AC.

Then if θ be the angle which DE makes with the horizon, we shall have, by resolving horizontally and vertically and taking moments about G,

$$R\sin\alpha - R'\sin\beta = 0 \quad\ldots\ldots(1),$$
$$R\cos\alpha + R'\cos\beta = W \quad\ldots\ldots(2),$$
$$Ra\cos(\alpha+\theta) - R'a\cos(\beta-\theta) = 0 \quad\ldots\ldots(3);$$

from (1) and (3)

$$\frac{\cos(\alpha+\theta)}{\sin\alpha} = \frac{\cos(\beta-\theta)}{\sin\beta};$$

or, $\cot\alpha\cos\theta - \sin\theta = \cot\beta\cos\theta + \sin\theta$;

or, $2\tan\theta = \cot\alpha - \cot\beta$.

XXI. Let us solve the preceding problem by a different method. The directions of R and R' if produced must meet in a certain point P, and the direction of W produced must pass through this same point. Suppose them to do so; then in the triangle PDG, $GPD = \alpha$, (for it is the inclination of DP to the vertical, which must be the same as that of AB to the horizon,)

$$PDG = 90^0 - ADG = 90^0 - \alpha - \theta,$$

$$\therefore \frac{PG}{DG} = \frac{\sin PDG}{\sin GPD} = \frac{\cos(\alpha+\theta)}{\sin\alpha};$$

in like manner, from the triangle PEG,

$$\frac{PG}{EG} = \frac{\cos(\beta-\theta)}{\sin\beta};$$

but $DG = EG$;

$$\therefore \frac{\cos(\alpha+\theta)}{\sin\alpha} = \frac{\cos(\beta-\theta)}{\sin\beta}, \text{ as before.}$$

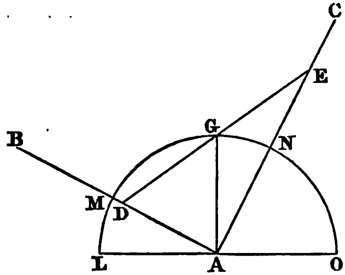
XXII. In the particular case in which the angle between the two planes is a right angle the result assumes a simpler form, and moreover gives rise to a remark upon a still different method of solving the problem. In this case, $\beta = 90^0 - \alpha$, and our equation becomes

$$2\tan\theta = \cot\alpha - \tan\alpha = \frac{1}{\tan\alpha} - \tan\alpha = \frac{1-\tan^2\alpha}{\tan\alpha};$$

$$\therefore \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{1}{\tan\theta} = \cot\theta = \tan(90^0 - \theta);$$

$$\text{or,}\quad 2\alpha = 90^0 - \theta, \qquad \text{or,}\quad \theta = 90^0 - 2\alpha.$$

Now suppose in this case that we make the beam DE to assume



all possible positions while its extremities move upon the planes AB, AC; then it is well known (and is easily proved) that the middle point G will trace out an arc MGN of a circle LGO, of which the centre is A, and the diameter DE, (LO is taken as the horizontal line). Then, since we may regard the weight of the beam as collected at G, the problem is precisely the same as that of finding the position of equilibrium of a particle upon the surface of the semicircle LGO. It is manifest that this position will be the highest point of the circle; therefore draw AG vertical, and the intersection of this line with the circle will fix the position of G; through G draw DGE so as to be bisected in G, (which may be done in several ways,) then DGE is the required position of the beam. And we have,

$$GAD = GDA = \alpha + \theta;$$

$$\text{but } GAD = 90^0 - DAL = 90^0 - \alpha;$$

$$\therefore \alpha + \theta = 90^0 - \alpha,$$

$$\text{or } \theta = 90^0 - 2\alpha, \text{ as before.}$$

The problem might be worked out in a similar manner for the more general case, but the locus of G would then be an ellipse instead of a circle, and the solution would not be so simple. This method of solution has the advantage of pointing out the character of the equilibrium of this problem with respect to *stability* (see p. 42); for it is evident that the equilibrium of a particle upon a smooth circle is unstable or only theoretically possible, therefore also the equilibrium of the beam DE is unstable. Practically the influence of friction will make the equilibrium possible, and that within considerably wide limits: let us investigate this case.

XXIII. Suppose the extremity D to be on the point of descending, then the friction at D will be in the direction BD, that at E will be in the direction EA, and the equations of the problem will be as follows, (see fig. p. 134):

$R\sin\alpha - \mu R\cos\alpha - R'\sin\beta - \mu R'\cos\beta = 0$ ....(1),

$R\cos\alpha + \mu R\sin\alpha + R'\cos\beta - \mu R'\sin\beta = W$........................(2),

$Ra\cos(\alpha+\theta) + \mu Ra\sin(\alpha+\theta) - R'a\cos(\beta-\theta) + \mu R a\sin(\beta-\theta) = 0$(3).

From (1) and (3) we have

$$\frac{\cos(\alpha+\theta)+\mu\sin(\alpha+\theta)}{\sin\alpha-\mu\cos\alpha} = \frac{\cos(\beta-\theta)-\mu\sin(\beta-\theta)}{\sin\beta+\mu\cos\beta}.$$

Let us simplify this equation by putting $\mu = \tan f$, then

$$\frac{\cos(\alpha+\theta-f)}{\sin(\alpha-f)} = \frac{\cos(\beta-\theta+f)}{\sin(\beta+f)};$$

$$\text{and } 2\tan\theta = \cot(\alpha-f) - \cot(\beta+f).$$

This gives us the value of θ when D is on the point of descending; if, on the other hand, D be on the point of ascending, we have only to change the sign of f, and, calling the value of θ for this case θ', we have

$$2\tan\theta' = \cot(\alpha+f) - \cot(\beta-f).$$

In the particular case in which $\beta = 90^0 - \alpha$,

$$2\tan\theta = \cot(\alpha-f) - \tan(\alpha-f),$$

$$\text{and } \theta = 90^0 - 2\alpha + 2f,$$

$$\text{similarly, } \theta' = 90^0 - 2\alpha - 2f,$$

$$\therefore \theta - \theta' = 4f,$$

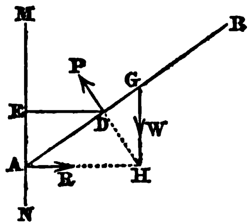
$$\text{or } f = \frac{\theta-\theta'}{4};$$

that is, the subsidiary angle f which we have introduced is one-fourth of the angle between the two extreme positions of the beam.

XXIV. Vary the preceding problem by taking one of the planes vertical; and let the planes be (1) smooth, (2) rough.

XXV. Vary it again by considering it as in XXIII, except that one plane shall be rough and the other smooth.

XXVI. A uniform beam rests upon a prop, with one extremity in contact with a smooth vertical wall. To find the conditions of equilibrium.



Let AB be the beam; G its middle point, or centre of gravity, W its weight, $2a$ its length, MN the wall, R the pressure of the wall upon the beam, which will be horizontal. D the prop, c its distance DE from the wall, P the pressure of the prop on the beam, which will be in a direction perpendicular to AB. Also let θ be the angle which AB makes with the horizon, or $\theta = ADE$.

Then resolving horizontally, and vertically, and taking moments about A,

$$R - P \sin\theta = 0 \ldots\ldots (1),$$
$$W - P \cos\theta = 0 \ldots\ldots (2),$$
$$Pc \sec\theta - Wa \cos\theta = 0 \ldots\ldots (3).$$

From (2) and (3) $\qquad c \sec\theta = a \cos^2\theta,$

$$\text{or } \cos^3\theta = \frac{c}{a},$$

$$\therefore \cos\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}},$$

which determines θ. It appears from the result that c must be less than a, as is represented in the figure, and as must manifestly be the case.

XXVII. We may solve this problem otherwise. In the triangle AED, the sides AE, ED, DA are respectively perpendicular to the directions of the forces R, W, P; hence they are in the same proportion as those forces, that is,

$$R : W : P :: AE : ED : DA,$$
$$:: \sin\theta : \cos\theta : 1.$$

This proportion is equivalent to equations (1) and (2). But we must obtain another equation in order to solve the problem; this is done from the consideration that the directions of three forces must meet in one point; let them meet in H. Then

$$\cos\theta = \frac{AD}{AH} = \frac{c \sec\theta}{a \cos\theta}, \text{ or } \cos^3\theta = \frac{c}{a} \text{ as before.}$$

XXVIII. Vary the preceding problem by taking the plane MN inclined at an angle α to the horizon.

If ϕ be the angle between the beam and MN, the result is

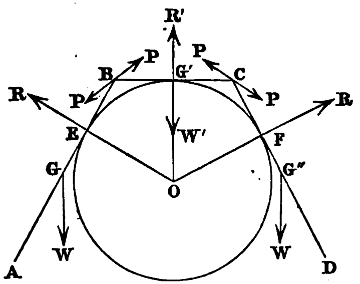
$$\sin^2\phi \cos(\alpha - \phi) = \frac{c}{a} \sin\alpha.$$

XXIX. Take the problem as before, with the exception of supposing a weight W' suspended from B.

XXX. If the wall be rough, find the limiting positions of equilibrium.

XXXI. Three uniform beams of length $2a$, $2b$, and $2a$ respectively, and of equal thickness, are loosely jointed together and suspended symmetrically from a fixed cylinder, of which the axis is horizontal and the radius greater than b, the middle beam resting upon the cylinder. Determine the pressure at the three points of contact; (1) when $2a$ is greater than b; (2) when it is less.

Let AB, BC, CD be the three beams, E, G', F the points of contact, G the middle point of AB. The figure will sufficiently explain the forces which act; it need only be remarked, that the



action between the two beams at B is assumed to be one force P acting in a direction determined by the angle θ which it makes with the horizon; we might have assumed a horizontal and vertical force, but have chosen to assume one force in a direction to be determined.

Let r be the radius of the cylinder, $EOG' = \alpha$, then the equations for the beam AB are

$$P\cos\theta - R\sin\alpha = 0 \quad\text{..................(1)},$$

$$P\sin\theta + R\cos\alpha = W \quad\text{.................(2)},$$

$$Rb - Wa\cos\alpha = 0 \quad\text{..................(3)};$$

for the beam BC,

$$R' - 2P\sin\theta = W' \quad\text{..................(4)};$$

the other two equations are identical; and those for CD are the same as for AB. And we have the geometrical relation,

$$b = r\tan\frac{\alpha}{2} \quad\text{..........................(5)}.$$

From (3)

$$R = \frac{Wa}{b}\cos\alpha = \frac{Wa}{b}\,\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{Wa}{b}\,\frac{r^2-b^2}{r^2+b^2}.$$

It will be observed that the problem is set with the condition that r is $> b$, and it is manifest that the beams AB, CD could not otherwise be in contact with the cylinder; in the expression just found for R, if r be less than b, R becomes negative; which shews that the beam AB could not be in contact unless held by a string, the tension of which would be a force in the opposite direction to that assigned in the figure to R.

Again, from (4),

$$\begin{aligned} R' &= W' + 2P\sin\theta, \\ &= W + 2W - 2R\cos\alpha, \quad \text{from (2)}, \\ &= W' + 2W - 2W\frac{a}{b}\cos^2\alpha, \\ &= W' + 2W\left\{1 - \frac{a}{b}\cdot\left(\frac{r^2-b^2}{r^2+b^2}\right)^2\right\}. \end{aligned}$$

Since the beams are of equal thickness we must have

$$W' : W :: b : a;$$

$$\therefore R' = W\left\{\frac{b}{a} + 2 - \frac{2a}{b}\left(\frac{r^2-b^2}{r^2+b^2}\right)^2\right\}.$$

Let us simplify these expressions by supposing that $r = 2b$,

$$\text{then } R = \frac{3a}{5b}W, \quad R' = W\left(\frac{b}{a} + 2 - \frac{18a}{25b}\right).$$

Still further, suppose that $a = 2b$,

$$\text{then } R = \frac{6}{5}W, \quad R' = \left(\frac{1}{2} + 2 - \frac{36}{25}\right)W = \frac{53}{50}W.$$

XXXII. The case in which $2a$ is $< b$ we shall leave to the student's own ingenuity, merely remarking that he will find the mechanical equations, with one exception, to be the same, and that he must substitute a new geometrical relation for equation (5). No step can be conveniently taken for the solution of the problem beyond writing down the appropriate equations.

XXXIII. Two equal and similar rough beams are fastened together by two of their extremities, so as to include a given angle and together to form one rigid piece; the piece thus formed is balanced upon a fixed rough horizontal cylinder. Determine the limiting positions of equilibrium, the coefficient of friction being $\tan f$.

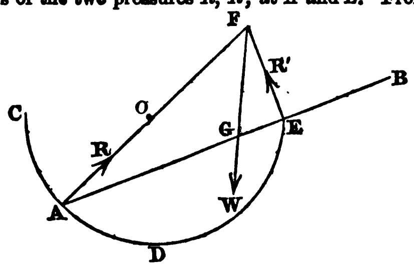
The student may, if he please, simplify this problem by taking the angle between the beams a right angle, and each of the beams equal in length to the diameter of the cylinder.

XXXIV. A person suspended in a balance, of which the arms are equal, thrusts his centre of gravity out of the vertical by means of a rod fixed to the further extremity of the beam of the balance, the direction of the rod passing through his centre of gravity: given that the rod and the line from the nearer end of the beam of the balance to his centre of gravity make angles α, β with the vertical, shew that his apparent and true weights are in the ratio $\sin(\alpha+\beta) : \sin(\alpha-\beta)$.

XXXV. A uniform beam is placed in a fixed smooth hemispherical bowl, the diameter of which is less than the length of the beam; find the position of equilibrium.

We will leave this problem to be treated by the student according to the general method of resolution of forces and of moments, and will insert a geometrical solution.

Let AB be the beam, CDE the bowl, O its centre; draw AOF, and EF perpendicular to AB, to meet in F; these will be the directions of the two pressures R, R', at A and E. From F draw



a vertical line, which must pass through G, the middle point of AB, and be the line of action of its weight W. Let r be the radius of the sphere, $AB=2a$, θ the angle which AB makes with the vertical, or $FGE=\theta$.

Then $OAE=OEA=90^\circ-\theta$. Also since AEF is a right angle, a semicircle described upon AF will pass through E, $\therefore AO=OF$, or $AF=2r$.

Now from the triangle FAG, we have

$$\frac{AG}{AF}=\frac{\sin AFG}{\sin AGF},$$

$$\text{or } \frac{a}{2r}=\frac{\sin(\theta-90^\circ+\theta)}{\sin\theta}=-\frac{\cos 2\theta}{\sin\theta};$$

$$\therefore \cos 2\theta + \frac{a}{2r} \sin \theta = 0,$$

$$\text{or } \sin^2 \theta - \frac{a}{4r} \sin \theta - \tfrac{1}{2} = 0,$$

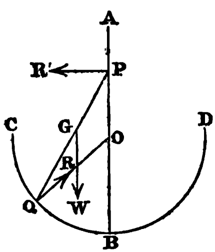
a quadratic for determining θ.

The student may perhaps be puzzled by obtaining a quadratic, which must have of necessity two roots, when apparently one answer only was required to the problem: by solving the equation, (or by observing the sign of its last term,) he will perceive that one root of the equation is positive and the other negative, and since the angle which he requires is manifestly less than 180^0 he will know that the positive root is that which he seeks. The full explanation however of the existence of the negative root would lead us into difficult considerations, which are better for the present omitted.

XXXVI. Consider the preceding problem, taking account of friction.

XXXVII. An aperture is made at the extremity of a horizontal diameter of a fixed hollow sphere, and a rod without weight and of greater length than the diameter of the sphere inserted; given the length of the rod, determine the conditions of equilibrium when it is acted upon by a given horizontal force, at the extremity which lies without the sphere.

XXXVIII. AOB is a fixed vertical rod in a fixed hemispherical bowl CBD, of which the centre is O. PQ is a uniform rod resting upon the bowl at Q and upon AOB at P: to find the position of equilibrium.



Join QO, this will be the direction of the pressure of the surface of the bowl upon the rod at Q, which call R. The pressure R' of the rod AOB upon PQ at P will be horizontal. The weight W of PQ will act parallel to AOB at the middle point G of PQ. Let $QPB=\theta$, $QOB=\phi$, $BO=r$, and $PQ=2a$.

Then resolving horizontally and vertically, and taking moments about G, we have

$$R' - R\sin\phi = 0 \ldots\ldots\ldots\ldots (1),$$

$$W - R\cos\phi = 0 \ldots\ldots\ldots\ldots (2),$$

$$R'a\cos\theta - Ra\sin(\phi - \theta) = 0 \ldots. (3).$$

And we have the geometrical relation,

$$\frac{2a}{r} = \frac{\sin\phi}{\sin\theta} \quad\text{.........(4).}$$

Now from (1) and (3) there results

$$\begin{aligned}\sin\phi\cos\theta &= \sin(\phi - \theta)\\ &= \sin\phi\cos\theta - \sin\theta\cos\phi;\end{aligned}$$

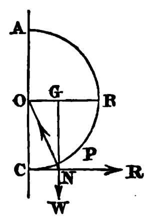
$$\therefore \sin\theta\cos\phi = 0.$$

Hence either $\theta = 0$, or $\phi = 90^0$; the latter is not possible, because (2) would give us $W = 0$; hence $\theta = 0$ is the only solution, that is, the rod PQ must stand vertically, in which case it is manifest that there will be equilibrium.

Consequently there is no such position of equilibrium as that which we have represented in the figure; and a little consideration would have pointed this out to us at first, for it is evident that the directions of the three forces W, R, R' cannot pass through the same point, which is an essential condition of equilibrium.

XXXIX. A ladder being placed against a perfectly smooth wall, find the smallest angle of elevation for which equilibrium is possible, the coefficient of friction between the ladder and ground being given.

XL. A hemisphere ABC being placed in contact with a smooth vertical wall, as in the figure; to find the nearest point to C at which a smooth prop being placed, the hemisphere will be at rest.



Let O be the centre of the sphere; draw OGB horizontal, then it is evident that the centre of gravity of the hemisphere will be somewhere in OB: let it be G; the exact position of G is a matter of indifference so far as the method of investigation is concerned, but we may as well assume that which can be proved, namely, that $OG : OB :: 3 : 8$. Let W be the weight of the hemisphere which acts at G.

Again, if there be a prop at any point P, it is plain that the hemisphere cannot fall without twisting so as to separate itself from the wall; just before the final separation takes place, the lowest point C will be the only point in contact with the wall: hence we may suppose the action of the wall to produce a force R at the point C, and in the direction perpendicular to AC, i. e. in the horizontal direction.

Let the directions of W and R intersect in N. Then the third force arising from the pressure of the prop must also act through

N; but this force must also act through O, since it must be normal to the surface of the sphere; hence, joining NO, this must be the direction of the force, and the point in which NO meets the hemisphere will be the point required.

If we denote by θ the angle CON, then $\tan\theta = \frac{CN}{OC} = \frac{3}{8}$, or $\theta = 20^0\,33'$: and the angle θ thus found will entirely fix the position of the point required.

We have solved this problem by a geometrical construction, as being the neatest method of treatment. If however we had taken P as the point, and denoted COP by θ, and the pressure at P by R', we should have had the equations

$$R' \cos\theta = W \text{.................}(1),$$
$$R' \sin\theta = R \text{.................}(2),$$
$$R \,.\, OC = W \,.\, OG \text{...........}(3);$$

(1) and (2) give us

$$\tan\theta = \frac{R}{W},$$

and from (3)

$$\frac{R}{W} = \frac{OG}{OC} = \frac{3}{8},$$

$$\therefore \tan\theta = \frac{3}{8} \text{ as before.}$$

XLI. Consider the preceding problem on the supposition of the wall being rough.

XLII. If the wall be smooth and the prop be not at the nearest point possible to C, but at any other point P such that $COP = \alpha$, determine the pressure upon the wall and prop respectively.

XLIII. A cylinder lies upon two equal cylinders, all in contact, and having their axes parallel: and the lower cylinders rest on a horizontal plane: $\tan f$, $\tan f'$ are the coefficients of friction respectively between the cylinders, and between the cylinders and the plane. Supposing the points of contact all to slip at the same instant, find f and f'.

Let W be the weight of the upper cylinder, W' that of either of the lower ones, R the action between two cylinders, R' between either of the lower cylinders and the plane; α the angle which the line joining the centres of the upper and either of the lower cylinders makes with the vertical. (The student can supply the figure.) Then the equations of the problem will be as follows.

For the upper cylinder,

$$2R\cos\alpha + 2R\tan f\sin\alpha = W,$$

$$\text{or } R\cos(\alpha - f) = \frac{W\cos f}{2} \quad\text{.................(1).}$$

For either of the lower cylinders,

$$R\sin(\alpha - f) = \cos f\tan f' R' \quad\text{.................(2),}$$

$$R\cos(\alpha - f) = (R' - W')\cos f \quad\text{...............(3),}$$

$$R\tan f = R'\tan f' \quad\text{.......................(4).}$$

From these four equations we can eliminate R and R', and find f and f'. From (2) and (4), we have

$$\sin(\alpha - f) = \sin f;$$

$$\therefore \alpha - f = f, \text{ or } f = \frac{\alpha}{2}.$$

Again, from (1) and (3),

$$\frac{W}{2} = R' - W', \text{ or } R' = W' + \frac{W}{2},$$

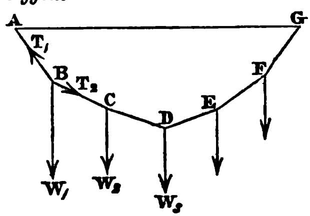
from (1) and (2)

$$\tan(\alpha - f) = \frac{2R'\tan f'}{W},$$

$$\text{or } \tan\frac{\alpha}{2} = \frac{2W' + W}{W}\tan f';$$

$$\therefore \tan f' = \frac{W}{2W' + W}\tan\frac{\alpha}{2}.$$

XLIV. A number of weights W_1, W_2, W_3, ... are connected by strings of given length, the extremities being attached to two points in the same horizontal line, so as to allow the weights to hang in a festoon below; to find the conditions of equilibrium.—*The Funicular Polygon.*



Let B, C, D, ... be the weights; A, G the points of suspension; $AB = a$, $BC = a_2$, &c. $AG = c$; also let the angles which AB, BC, ... respectively make with the vertical be θ_1 θ_2...; lastly, let T_1 T_2...... be the tensions of the strings.

Then the weight W_1 is kept in equilibrium by its own weight acting vertically, T_1 acting in the direction BA, and T_2 in the direction BC. Hence resolving horizontally and vertically we have,

$$T_1 \sin\theta_1 = T_2 \sin\theta_2 \text{.........(1)},$$
$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = W_1 \text{.........(2)}.$$

We shall have two similar equations for each weight; suppose there are n of them, then we shall have $2n$ equations. Let us consider how many unknown quantities there will be. There are $n+1$ strings, therefore there will be $n+1$ unknown forces $T_1\, T_2 \ldots T_{n+1}$, and $n+1$ unknown angles $\theta_1\, \theta_2 \ldots\ldots \theta_{n+1}$; on the whole then there will be $2n+2$ unknown quantities; but there are only $2n$ mechanical equations, therefore we must have two geometrical. One of these will be the following, which arises from the fact of AG being given,

$$a_1 \sin\theta_1 + a_2 \sin\theta_2 + \ldots\ldots + a_{n+1} \sin\theta_{n+1} = c. \quad \text{(A)}$$

The other is obtained thus; let D be the lowest point of the festoon, and let W_p be the weight which hangs at D, then the vertical distance of D from AG is

$$a_1 \cos\theta_1 + a_2 \cos\theta_2 + \text{\&c.} + a_p \cos\theta_p;$$

but the same vertical distance is also equal to

$$a_{p+1} \cos\theta_{p+1} + \ldots\ldots + a_{n+1} \cos\theta_{n+1};$$

$$\therefore a_1 \cos\theta_1 + \ldots\ldots + a_p \cos\theta_p = a_{p+1} \cos\theta_{p+1} + \ldots\ldots + a_{n+1} \cos\theta_{n+1}. \text{(B)}$$

Thus we have obtained the $2n+2$ equations required.

To complete the solution of the problem so far as it can be completed, let us write down equation (2) and all those of the same class: and to simplify the problem we will consider all the weights equal, then

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = W,$$
$$T_2 \cos\theta_2 - T_3 \cos\theta_3 = W,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_n \cos\theta_n - T_{n+1} \cos\theta_{n+1} = W;$$

subtracting the second of these equations from the first, we have

$$T_1 \cos\theta_1 - 2T_2 \cos\theta_2 + T_3 \cos\theta_3 = 0.$$

But by (1) $\qquad T_1 \sin\theta_1 = T_2 \sin\theta_2,$

and in like manner, $\qquad T_2 \sin\theta_2 = T_3 \sin\theta_3,$

$$\therefore T_1 \sin\theta_1 = T_2 \sin\theta_2 = T_3 \sin\theta_3 = \lambda \text{ suppose};$$

$$\therefore T_1 = \frac{\lambda}{\sin\theta_1}, \quad T_2 = \frac{\lambda}{\sin\theta_2}, \quad T_3 = \frac{\lambda}{\sin\theta_3}.$$

Hence our equation becomes

$$\frac{\lambda\cos\theta_1}{\sin\theta_1} - 2\frac{\lambda\cos\theta_2}{\sin\theta} + \frac{\lambda\cos\theta_3}{\sin\theta_3} = 0,$$

$$\text{or } \cot\theta_1 + \cot\theta_3 = 2\cot\theta_2.$$

The angles θ_1, θ_2, θ_3, therefore, and in like manner all the following angles, are such that their cotangents form an arith-

metical progression. In this arithmetical progression it is evident that the terms must be decreasing, (since the angles increase, and therefore their cotangents diminish;) hence the terms will at length become negative, that is, the angle will be greater than a right angle, and these will correspond to the strings between the lowest weight and the point G.

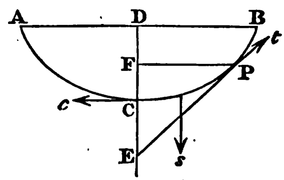
The set of equations similar to (1) express that the horizontal tension of all the strings is the same; a result which might have been anticipated. And this same thing is true of a chain or cord suspended from its two extremities, and so forming what is called a *catenary;* that is to say, the horizontal tension is the same at all points of the chain or cord.

XLV. Two equal weights are suspended from two points in the same horizontal line, which are two feet apart, by means of three threads which are respectively 1, 1 and 2 feet long; prove that the three acute angles which the threads make with the horizontal line are together equal to two right angles.

XLVI. If in the preceding problem the connecting threads be each one foot long, determine the angles which they make with the horizon, and their tensions.

XLVII. Having alluded to the form assumed by a cord or chain suspended from its two extremities, it may be as well to make one or two more remarks upon the same subject. The form of the curve we shall not be able to investigate, without recourse to a higher portion of the pure mathematics than we suppose the student to have at his command; nevertheless certain properties of the curve may be investigated without difficulty.

Let ACB be a uniform cord, hanging from two points AB in the same horizontal line; D the middle point between A and B; DC a vertical line. Then it is evident, that the portion AC of the cord on one side of DC will be precisely similar to the portion BC on the other side, and C will be the lowest point.



Now how are we to apply our principles of equilibrium, which have been investigated for a *particle* and for a *rigid body,* to the case of a *cord* which is neither the one nor the other? The method adopted is as follows.

Let P be any point in the cord, and let the length of $CP=s$. Then since the cord is at rest, it will not affect the equilibrium if we *suppose* the portion CP to become rigid; suppose, for instance,

CP to be impregnated with some fluid which solidifies, so that CP hangs like a wire with cords at its extremities. The rigid piece CP will be held in equilibrium by the vertical force of its own weight, which will be proportional to s and will act through its centre of gravity, and by the tensions of the cord at C and P; now these tensions will be in the directions of the tangents of the curve at those points, therefore the tension at C will be evidently horizontal, and that at P will be in the direction of EP, suppose. We have already explained that the horizontal tension will be the same at all points of the *catenary*, therefore the tension at C which is wholly horizontal will be equal to this constant horizontal tension; let c be the length of cord the weight of which would be equal to this tension; then as the weight of CP is to s, so is the tension at C to c. In like manner, denote the unknown tension at P by t, where t is in fact the length of cord whose weight would produce that tension.

Now draw PF horizontal, that is, perpendicular to DCE. Then CP is kept in equilibrium by the three forces s, c, t which act in directions respectively parallel to FE, PF, EP; therefore by the Triangle of Forces,

$$s : c : t :: FE : PF : EP.$$

Let $FEP = \theta$, then

$$\frac{s}{c} = \frac{FE}{PF} = \cot\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{and } \frac{t}{c} = \frac{EP}{PF} = \operatorname{cosec}\theta, \text{ or } t^2 = c^2 + s^2 \ldots(2).$$

We cannot proceed any further in the solution; but it will be seen that we have obtained two important results.

For (1) gives us the law according to which the direction of the tangent of the catenary changes as we proceed from the lowest point; and it will be interesting to notice how this result may be obtained from the corresponding equation in the investigation of the funicular polygon.

Now we may regard a uniform cord as a series of equal weights connected by strings of indefinitely small length, and then the direction of the string joining any two weights will be the direction of the tangent at the corresponding point of the catenary. Let then the cord s be divided into any number (p) of equal parts; then the weight of each portion will be measured by $\frac{s}{p}$. Now resuming equations (1) and (2) of the funicular polygon (p. 145), we have

$$T_1 \sin\theta_1 = T_2 \sin\theta_2 = c,$$

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = \frac{s}{p};$$

$$\therefore c \cot \theta_1 - c \cot \theta_2 = \frac{s}{p},$$

$$\text{or, } \cot \theta_1 - \cot \theta_2 = \frac{1}{p} \cdot \frac{s}{c},$$

$$\text{in like manner, } \cot \theta_2 - \cot \theta_3 = \frac{1}{p} \cdot \frac{s}{c},$$

$$\dots\dots\dots\dots = \dots\dots$$

$$\text{and } \cot \theta_p - \cot \theta_{p+1} = \frac{1}{p} \cdot \frac{s}{c};$$

therefore, adding all these equations together,

$$\cot \theta_1 - \cot \theta_{p+1} = p \times \frac{1}{p} \cdot \frac{s}{c} = \frac{s}{c}.$$

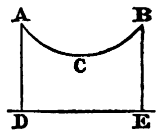
But at the lowest point of the catenary $\theta_{p+1} = 90^0$, or $\cot \theta_{p+1} = 0$; and θ_1 is the angle which we have called θ in the investigation of the catenary;

$$\therefore \cot \theta = \frac{s}{c} \text{ as before.}$$

Again, equation (2) gives us the law according to which the tension of the cord in the catenary varies from point to point. This may be in like manner deduced from the equation (1) of the funicular polygon.

XLVIII. The following is an example of the application of the preceding investigation.

A chain of given weight is suspended from two equal vertical posts of given height, and the angle which the chain makes with either post at the point of suspension is observed. To find the moment of the force which tends to overturn one of the posts.



If h be the height of one of the posts, then, according to our previous notation, ch is the moment required. To determine c, we observe, that if W be the whole weight of the chain, and a the angle which the chain makes with either post, then

$$\frac{W}{c} = \cot a,$$

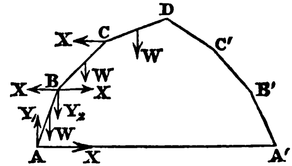
$$\therefore c = W \tan a,$$

and $Wh \tan a$ is the moment required.

It may be remarked here, that although a cord may be stretched so as to be for all practical purposes horizontal, as in the case of a pianoforte string, yet it never can be so accurately. In the preceding investigation a cannot be 90^0 unless either $W = 0$ or $c = \infty$.

XLIX. Analogous to the problem of the funicular polygon is that of the conditions of equilibrium of a system of beams, resting one upon another so as to form a kind of arch. The essential distinction between the two problems is, that in the case of the beams we have a series of equations of *moments*, which do not occur in the other.

To simplify the problem we will suppose the beams to be all equal and uniform, and that there are an even number of them. Then the arrangement will be such as is represented in the figure.



Let us consider the forces which act. Upon each beam there will be the vertical force W, its weight, acting at its middle point. At each extremity of each beam there must be a force, which it will be convenient to consider as resolved into a horizontal and a vertical part; this will be convenient, because it is evident that all the horizontal parts must be the same throughout the system, and we may therefore denote them by one common symbol X: let Y_1, Y_2, Y_3,...... be the upward vertical pressures at the lower extremities of the 1st, 2nd, 3rd,...... beams respectively, reckoning from the lowest; and let θ_1, θ_2, θ_3, ... be the angles which the same make with the horizon. Also let there be $2n$ beams, and let the length of each be a.

Then the mechanical equations for the n beams on the left of the vertical through the highest point D will be as follows:

$$\left.\begin{array}{l} Y_1 - Y_2 = W, \\ Y_1 a \cos\theta_1 + Y_2 a \cos\theta_1 = 2Xa \sin\theta_1 \end{array}\right\}\ldots\ldots\ldots(1).$$

$$\left.\begin{array}{l} Y_2 - Y_3 = W, \\ Y_2 a \cos\theta_2 + Y_3 a \cos\theta_2 = 2Xa \sin\theta_2 \end{array}\right\}\ldots\ldots\ldots(2).$$

....

$$\left.\begin{array}{l} Y_n = W, \\ Y_n\, a \cos\theta_n = 2Xa \sin\theta_n \end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(n).$$

It will be observed that there is no vertical pressure upon the upper end of the last or n^{th} beam. Thus we have $2n$ equations involving $2n+1$ unknown quantities, viz. X, Y_1, Y_2,...Y_n, θ_1, θ_2,...θ_n. We must have *one* geometrical equation, which will result from the fact of the distance AA' between the points of support of the system being given; call it $2c$, then we have

$$a\cos\theta_1 + a\cos\theta_2 + \ldots + a\cos\theta_n = c\ldots\ldots(n+1).$$

We can proceed a little further in the solution of the problem.

We have $$Y_n = W,\ Y_{n-1} - Y_n = W;$$
$$\therefore\ Y_{n-1} = 2W,$$
in like manner $Y_{n-2} = 3W$, and so on.
Lastly, $Y_1 = nW$.

This result might have been foreseen, since the point of support A bears the whole weight of the beams, that is, the vertical pressure upon it is nW.

Again, we have

$$2X \tan\theta_1 = Y_1 + Y_2 = nW + (n-1)W = (2n-1)W,$$
$$2X \tan\theta_2 = Y_2 + Y_3 = (n-1)W + (n-2)W = (2n-3)W,$$
....
$$2X \tan\theta_n = Y_n = W;$$
$$\therefore \frac{\tan\theta_1}{\tan\theta_2} = \frac{2n-1}{2n-3},\ \frac{\tan\theta_2}{\tan\theta_3} = \frac{2n-3}{2n-5},\ \&c.,\quad \frac{\tan\theta_{n-1}}{\tan\theta_n} = 3.$$

We cannot carry the solution any further, because of the unmanageable character of the equation $(n+1)$.

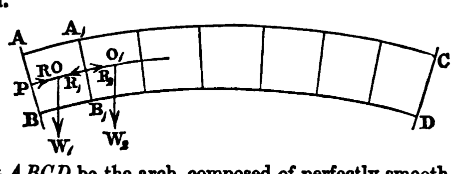
L. Three uniform beams rest upon abutments, as in the preceding problem. Write down the equations necessary for the solution of the problem.

LI. Let AB, BC, CD be the three beams; and let $AB = BC$, and $AD = CD = 2AB$; then if a circle be described through B and C and touching the horizontal line through B, it will pass through the point of intersection of AB, DC produced.

LII. Two equal beams rest upon a smooth horizontal plane, bearing upon each other at their highest points and having their lower extremities connected by a string of given length; find the tension of the string.

In common roofs the lower extremities of the principal rafters are frequently connected by a beam called a *tie-beam;* the result of the preceding problem will point out the use of the tie-beam, in preventing any spreading of the walls in consequence of the outward pressure caused by the weight of the roof.

LIII. Supposing the surfaces of the stones, or *voussoirs*, of an arch to be perfectly smooth, to determine the conditions of equilibrium.



Let $ABCD$ be the arch, composed of perfectly smooth stones,

which press against each other, and rest upon solid masonry at AB and CD.

Consider the equilibrium of the first stone, or voussoir, ABB_1A_1, the weight of which call W_1, which acts through the centre of gravity. Now the pressures of the voussoir upon the stone-work at AB will be equivalent to some one pressure R acting at a point P perpendicularly to AB. Let the direction of this pressure meet that of W_1 in O; then the pressure between the first and second voussoirs must be a pressure R_1 passing through O: therefore if we draw OP_1 perpendicular to A_1B_1, this will be the direction of R_1. In like manner we may determine the direction of the action between the second and third voussoirs, and so on.

Let $\theta\theta_1\theta_2$...be the angles which AB, A_1B_1, A_2B_2, ... make with the vertical. Then we shall have the following equations,

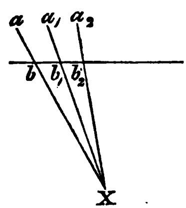
$$\left.\begin{array}{l} R\cos\theta - R_1\cos\theta_1 = 0, \\ R\sin\theta - R_1\sin\theta_1 = W_1 \end{array}\right\}\cdots\cdots\cdots\cdots(1)$$

$$\left.\begin{array}{l} R_1\cos\theta_1 - R_2\cos\theta_2 = 0, \\ R_1\sin\theta_1 - R_2\sin\theta_2 = W_2 \end{array}\right\}\cdots\cdots\cdots\cdots(2),$$

&c. &c.

The analogy of these equations to those of the funicular polygon will at once be noticed.

We shall not pursue this investigation further, but will point out a simple geometrical construction by means of which we can determine, if the directions of the joints of the voussoirs be given, the relations of the weights, and *vice versâ*.

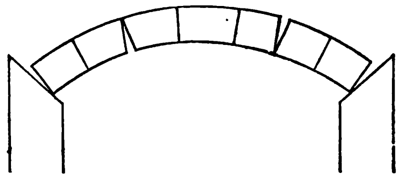


(1) Let the directions AB, A_1B_1, &c. be given.

Through any point X draw Xa, Xa_1, Xa_2 ... parallel to AB, A_1B_1, A_2B_2, ... and draw any horizontal line cutting the above lines in bb_1b_2 ... respectively. Then the straight lines bb_1, b_1b_2, ... will be proportional in length to the weights $W_1\,W_2$... The student will easily supply the proof.

(2) Let the weights W_1, W_2, W_3, ... be given. The direction of the face of the abutment AB is arbitrary, let it be abX; and through any point X in it, draw a straight line parallel to the face of the other abutment (not represented in the figure). Draw a horizontal line cutting these two, and divide the intercepted part in b_1, b_2, ... so that bb_1, b_1b_2 ... shall be in the same proportion as W_1, W_2, ... Join Xb_1, Xb_2, ...; these will be the required directions of the joints.

The preceding investigation, though useful as an illustration of statical principles, is of no practical value in the construction of arches. For the hypothesis of the smoothness of the voussoirs is one which does not hold even approximately; the force of friction is necessarily great, and may be made as large as we please. An arch constructed upon the preceding principles would stand; but with this peculiarity, that any additional pressure to any one of the voussoirs would cause that voussoir to sink and others to rise, until the *line of pressure* had adjusted itself to the altered circumstances. But when the friction is such as to prevent the voussoirs from sliding upon each other, the arch cannot give way except by *breaking:* and a little con-



sideration will shew that it must break into at least *three* portions, and the fractures must be alternately in the *extrados* and *intrados*, or external and internal curve of the arch. The true theory of the arch therefore is much more nearly connected with that of the equilibrium of beams, than that of the theoretical problem of an arch built with smooth voussoirs.

LIV. A string having its extremities fixed to the ends of an uniform rod, of weight W, passes over four tacks, so as to form a regular hexagon; the rod, which is horizontal, being one of the sides; find the tension of the string and the vertical pressure on each tack. Shew also that the rod cannot hang in any other than a horizontal position.

LV. If one of the highest tacks be removed, compare the vertical pressure upon the other highest tack with its value as obtained in the preceding problem.

LVI. Still further; of the three remaining tacks remove that which is furthest from the highest, and again calculate the vertical pressure upon the highest tack.

LVII. A uniform beam rests with one end upon a given smooth inclined plane, and is supported upon a prop which is at a given distance from the inclined plane; determine the limits between which the length of the beam must lie in order that equilibrium may be possible.

LVIII. If in the preceding problem the beam be too long for equilibrium to be possible, and instead of merely resting upon the inclined plane it be therefore attached at a given point by means of a hinge, find the direction and magnitude of the strain upon the hinge.

LIX. A ladder rests in a given position against a smooth wall with its foot upon a rough pavement; determine the weight which must be placed at its foot to prevent it from sliding; the coefficient of friction for the ladder and the weight being both given.

LX. An equilateral triangle of given weight is supported in a horizontal position by three equal threads; the strength of the threads is such, that two of them would just support the triangle; determine the greatest weight which can be placed upon the triangle without breaking the threads.

LXI. A ladder rests against a wall; given the weight of the ladder, the angle which it makes with the horizon, and the coefficient of friction both for the ground and also for the wall; find how high a man of given weight can ascend without causing the ladder to slip.

LXII. A uniform string hangs in equilibrium over two pegs (not in the same horizontal line), shew by general reasoning that the two extremities must be in the same horizontal line.

First suppose that we have four pegs A, B, C, D, in the same vertical plane, of which A is vertically over C, and B vertically over D; and let C and D be in the same horizontal line, but A and B not in the same horizontal line. Take an endless string, and suspend it upon these four pegs; then it will arrange itself in some position of equilibrium, forming a festoon or catenary from A to B, and another from C to D; the catenary CD will be perfectly symmetrical with respect to C and D, and therefore will produce precisely the same tension at C as at D: whatever horizontal tension there may be at C and D will be counteracted by the pegs, and the vertical tension will be equivalent to two equal weights hung upon the strings AC, and BD: these equal weights may be represented by two equal pieces of string CE, DF; hence equilibrium will still subsist if we remove the pegs C and D, and the catenary CD, and instead thereof add the equal lengths of string CE and DF. We have now the string $EABF$ in equilibrium upon the two pegs A, B, with its extremities in the same horizontal line; and conversely it is not difficult to shew, that the string cannot be in equilibrium unless its extremities are in the same horizontal line.

Suppose A and B to be in the same horizontal line, then it is easy to calculate the length of AE and BF in order that there may be equilibrium. For we have already proved the general formula for the tension $t^2=s^2+c^2$. Hence if $2s$ be the whole length of the string we must have $AE^2=s^2+c^2$.

LXIII. A hemisphere is placed with its base in contact with a smooth wall, and it is moveable about a hinge at its lowest point; a string fixed to a point in the wall vertically over the hinge carries a weight, which presses the string against the hemisphere, and so preserves equilibrium: find the smallest weight which will answer the purpose. The distance of the point of attachment of the string from the hinge may be taken to be three times the radius.

When the system is in equilibrium, the student will observe that the string may be supposed to be fastened to the hemisphere at the two extreme points of the arc of contact, and therefore the action of the string will be reduced to that of two forces at those two points, each equal to the suspended weight.

LXIV. A sphere rests upon two inclined planes; find the pressure sustained by each.

LXV. A sphere rests upon a fixed horizontal plane: two equal rods, connected together at their higher ends by a hinge, rests symmetrically upon the sphere, their lower ends touching without pressing the horizontal plane. Find the inclination of either rod to the vertical.

LXVI. An isosceles right-angled triangle rests in a vertical plane with the right angle downwards, between two pegs at a distance a from each other in a horizontal line; determine the positions of equilibrium.

LXVII. If G be the centre of gravity of a triangle ABC, three forces in the direction of and proportional to GA, GB, GC, will keep a particle at G at rest.

LXVIII. A hemisphere is supported by friction with its curved surface resting upon a horizontal and in contact with a vertical plane; find the limiting position of equilibrium.

LXIX. AB is a rod capable of turning freely about its extremity A, which is fixed, CD is another rod equal to $2AB$, and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force acts at C in the direction CA, find the force which must be applied at D in order to produce equilibrium.

LXX. A uniform beam is hung from a fixed point by two unequal strings of given lengths attached to its extremities: compare the tension of each string with the weight of the beam.

LXXI. If a set of forces, acting at the angular points of a plane polygon be represented in magnitude and direction by the sides, taken in order, shew that the tendency to turn the body about an axis perpendicular to the plane of the polygon is the same through whatever point of the plane the axis passes.

LXXII. A triangular board of given weight rests in equilibrium with its base on a horizontal plane sufficiently rough to prevent all sliding. A force acts upon it in its own plane and in a given line through the vertex and without the triangle; find the limits between which the magnitude of the force must lie if the equilibrium is preserved.

LXXIII. Two equal circular disks with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the line bisecting the angle. Find the radius of the least disk which may be pressed between them without causing them to separate.

LXXIV. If two scales, one containing a weight P and the other a weight Q, be suspended by a string over a rough sphere, and if Q be on the point of descending, then the weight $\dfrac{Q^2 - P^2}{P}$ put into the opposite scale will make that scale to be on the point of descending.

LXXV. A uniform and straight plank rests with its middle point upon a rough horizontal cylinder which is fixed, their directions being perpendicular to each other. Find the greatest weight that can be put upon one end of the plank without causing it to slide from the cylinder.

LXXVI. A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string the length of which is equal to a side of the square: shew that the distances of three of its angular points from the wall are as 1, 3, and 4.

LXXVII. A heavy equilateral triangle, hung up on a smooth peg by a string the ends of which are attached to two of its angular points rests with one of its sides vertical; shew that the length of the string is double the altitude of the triangle.

LXXVIII. One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests upon another smooth plane, inclined to the horizon

at an angle (a); the string passing over a pully at the top of the inclined plane hangs vertically, supporting a weight (P). Shew that the beam will rest in any position if a certain relation hold between P, W, and a.

LXXIX. A cylinder, the base of which is in contact with a smooth vertical plane, is supported by a string fastened to it at a point of its curved surface whose distance from the vertical plane is x. Shew that x must be intermediate in value to $b - 2a \tan \theta$, and b, where $2b$ is the altitude of the cylinder, a the radius of the base, and θ the angle which the string makes with the vertical.

LXXX. A flat board in the form of a square is supported upon two smooth props with its plane vertical; investigate an equation for determining its positions of equilibrium, the distance between the props being equal to half a side of the square.

The equation required is $\cos \theta - \sin \theta = \cos (2\theta + a)$, where a is the angle which the straight line joining the props makes with the horizon, θ that which one side of the square makes with the same.

LXXXI. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each project as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

LXXXII. A uniform slender rod passes over the fixed point A and under the fixed point B, and is kept at rest by the friction at the points A and B; determine the limiting position of equilibrium.

LXXXIII. Four uniform slender rods AB, BC, CD, DA, rigidly connected, form the sides of a quadrilateral figure, such that the angle A is a right angle, and the points B, C, D are equidistant from each other; when the whole is suspended at the angle A, determine the position of equilibrium.

LXXXIV. A uniform bent lever, the arms of which are at right angles to each other, is just capable of being inclosed in the interior of a smooth spherical surface; determine the position of equilibrium.

It will be seen that the plane of the lever must be vertical, so that the directions of the forces will all lie in one plane.

LXXXV. Two unequal weights, connected by a straight rod without weight, are suspended by a thread of given length, fastened at the extremities of the rod, and passing over a fixed point; determine the position of equilibrium.

LXXXVI. A piece of uniform wire is formed into a triangle; find the distance of the centre of gravity of the periphery of the triangle from each of the sides; and shew that if x, y, z be the three distances, and r the radius of the inscribed circle, then

$$4xyz - r^2(x + y + z) = r^3.$$

LXXXVII. A uniform rod of length l is cut into three portions a, b, c; and these are formed into a triangle. When the triangle is placed in unstable equilibrium, resting with its plane vertical, one of its angular points being supported upon a smooth horizontal plane, find the angles which the uppermost side makes with the horizon; and shew that, if α, β, γ, be the three angles corresponding to the several cases of a, b, c being the uppermost side, then

$$(l + a) \tan \alpha + (l + b) \tan \beta + (l + c) \tan \gamma = 0.$$

LXXXVIII. A smooth body in the form of a sphere is divided into hemispheres and placed with the plane of division vertical upon a smooth horizontal plane; a string loaded at its extremities with two equal weights hangs upon the sphere, passing over its highest point and cutting the plane of division at right angles; find the least weight which will preserve equilibrium.

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